Dispersive Quantization — the Talbot Effect

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Introduction to Partial Differential Equations

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Dispersion

Definition. A linear partial differential equation is called dispersive if the different Fourier modes travel unaltered but at different speeds.

Substituting

$$u(t,x) = e^{\,\mathrm{i}\,(k\,x - \omega\,t)}$$

produces the dispersion relation

$$\omega = \omega(k)$$

relating frequency ω and wave number k.

Phase velocity: $\quad c_p = \dfrac{\omega(k)}{k}$

Group velocity: $\quad c_g = \dfrac{d\omega}{dk} \qquad$ (stationary phase)

A Simple Linear Dispersive Wave Equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

$\Longrightarrow$ linearized Korteweg–deVries equation

Dispersion relation: $\omega = k^3$

Phase velocity: $c_p = \dfrac{\omega}{k} = k^2$

Group velocity: $c_g = \dfrac{d\omega}{dk} = 3\,k^2$

Thus, wave packets (and energy) move *faster* (to the right) than the individual waves.

Linear Dispersion on the Line

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \qquad u(0,x) = f(x)$$

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$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k)\, e^{\,\mathrm{i}\,(k\,x + k^3\,t)}\, dk$$

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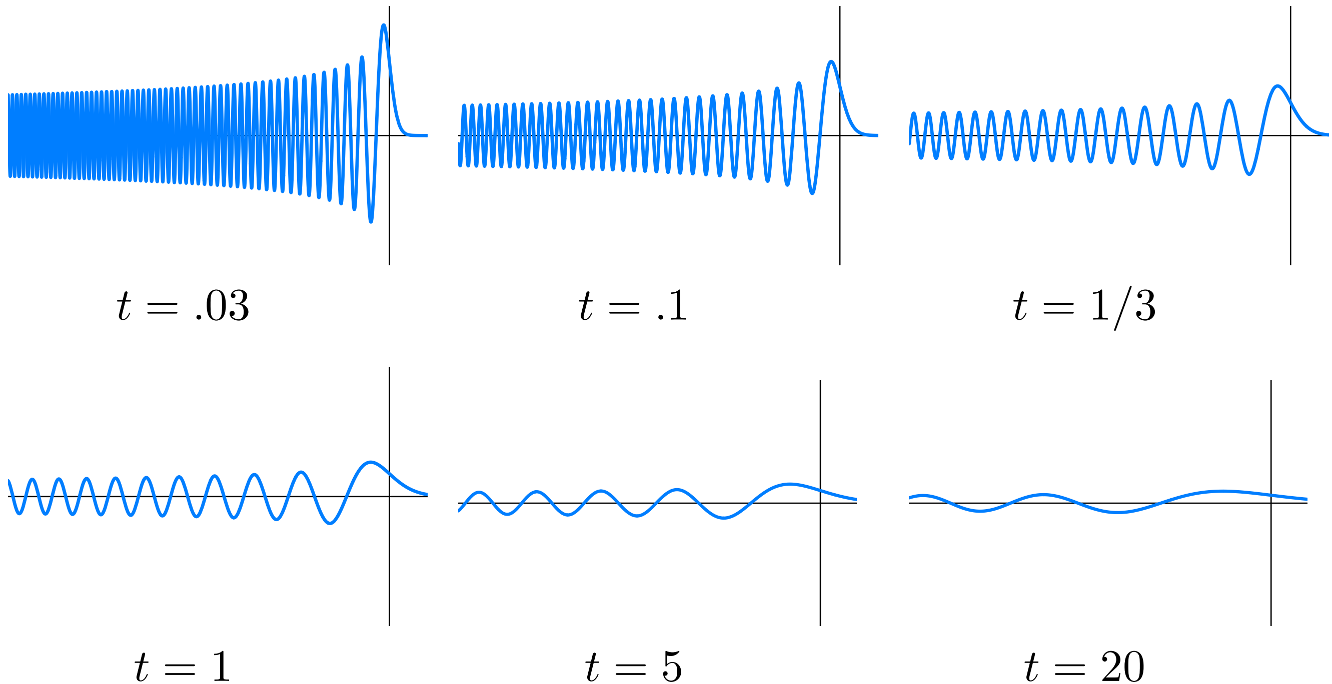
Fundamental solution $\qquad u(0,x) = \delta(x)$

$$u(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\,\mathrm{i}\,(k\,x + k^3\,t)}\, dk = \frac{1}{\sqrt[3]{3\,t}}\ \mathrm{Ai}\left(\frac{x}{\sqrt[3]{3\,t}} \right)$$


$t = .03$
$t = .1$
$t = 1/3$
$t = 1$
$t = 5$
$t = 20$

Linear Dispersion on the Line

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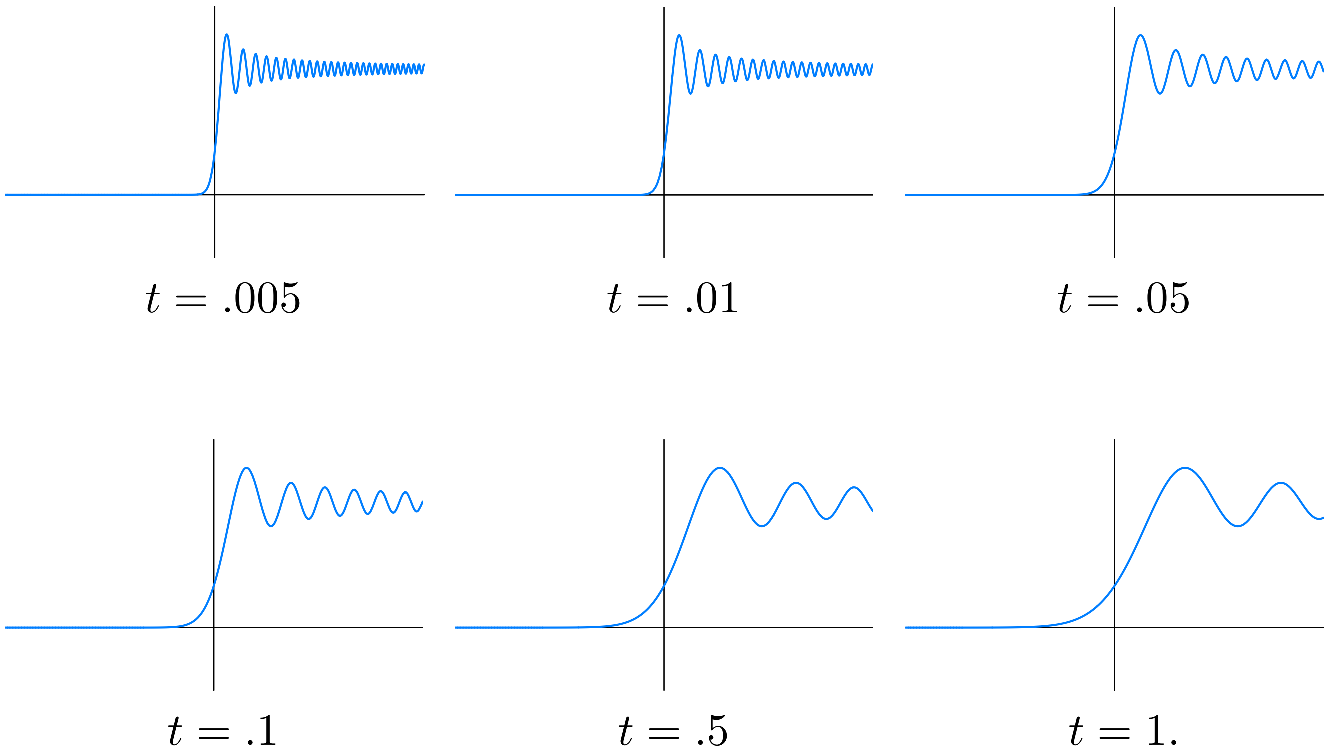
$$u(t,x) = \frac{1}{3} - H\left(-\frac{x}{\sqrt[3]{3\,t}}\right)$$

$$H(z) = \frac{z\,\Gamma\!\left(\frac{2}{3}\right)\,{}_1F_2\left(\frac{1}{3};\,\frac{2}{3},\,\frac{4}{3};\,\frac{1}{9}\,z^3\right)}{3^{5/3}\,\Gamma\!\left(\frac{2}{3}\right)\Gamma\!\left(\frac{4}{3}\right)} - \frac{z^2\,\Gamma\!\left(\frac{2}{3}\right)\,{}_1F_2\left(\frac{2}{3};\,\frac{4}{3},\,\frac{5}{3};\,\frac{1}{9}\,z^3\right)}{3^{7/3}\,\Gamma\!\left(\frac{4}{3}\right)\Gamma\!\left(\frac{5}{3}\right)}$$


$t = .005$
$t = .01$
$t = .05$
$t = .1$
$t = .5$
$t = 1.$

Periodic Linear Dispersion

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi) \qquad \frac{\partial^2 u}{\partial x^2}(t,-\pi) = \frac{\partial^2 u}{\partial x^2}(t,\pi)$$

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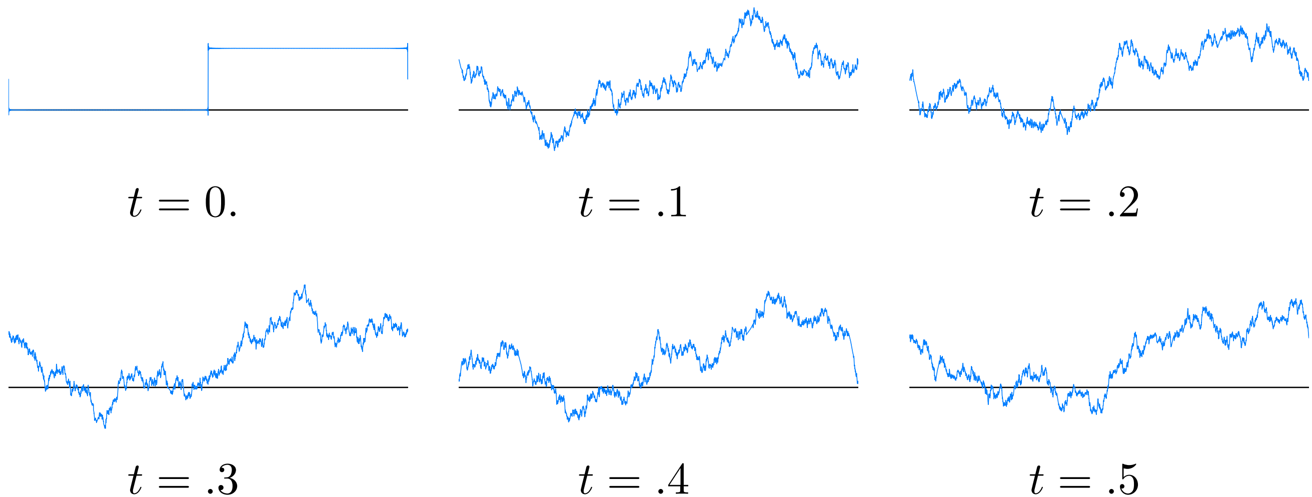
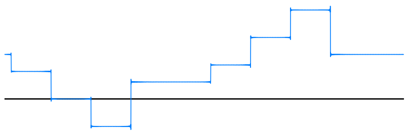
Step function initial data:

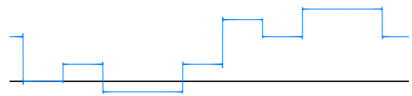
$$u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

$$u^\star(t,x) \;\sim\; \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin(\,(2\,j+1)\,x - (2\,j+1)^3\,t\,)}{2\,j+1}\,.$$

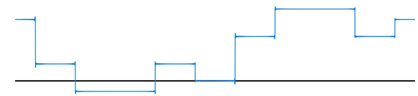

$t = 0.$
$t = .1$
$t = .2$
$t = .3$
$t = .4$
$t = .5$



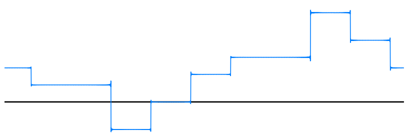
$t = \frac{1}{30}\pi$



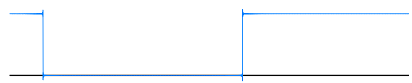
$t = \frac{1}{15}\pi$



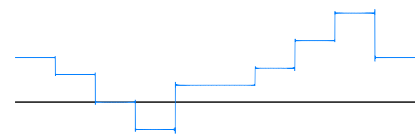
$t = \frac{1}{10}\pi$



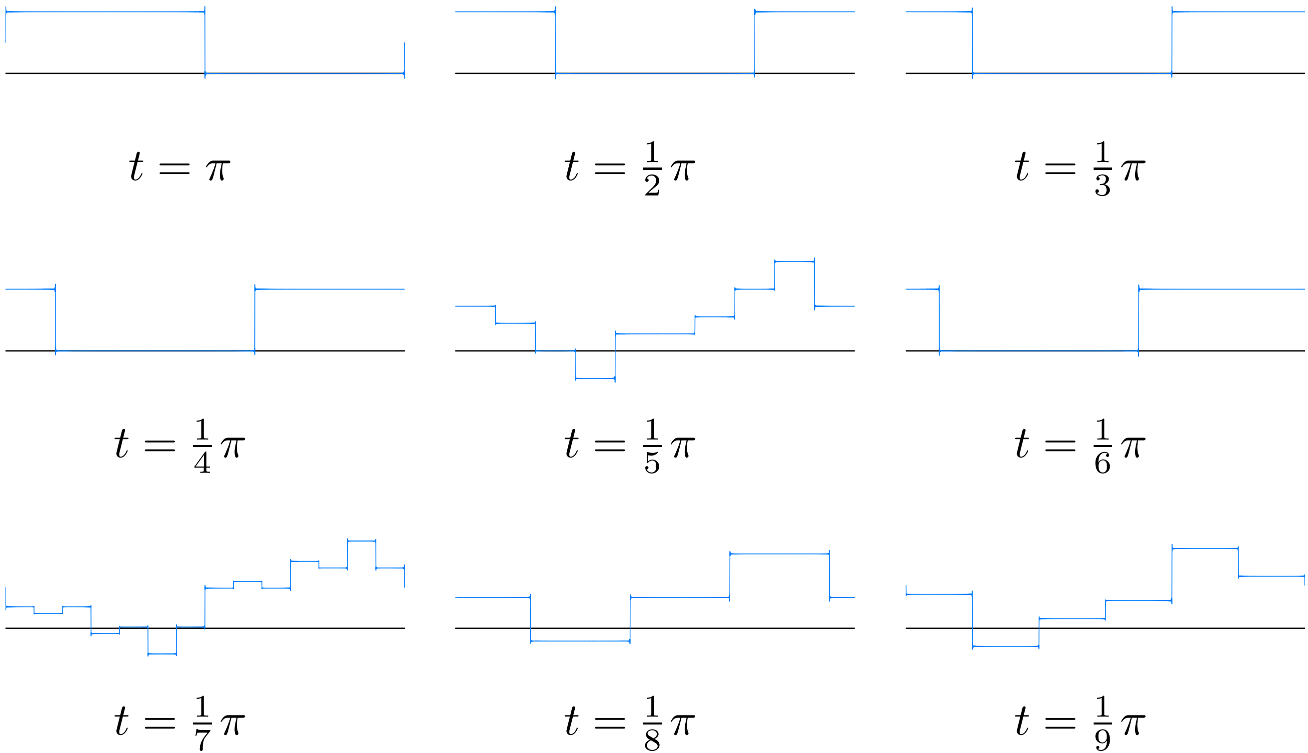
$t = \frac{2}{15}\pi$



$t = \frac{1}{6}\pi$



$t = \frac{1}{5}\pi$


$t = \pi$
$t = \frac{1}{2}\pi$
$t = \frac{1}{3}\pi$
$t = \frac{1}{4}\pi$
$t = \frac{1}{5}\pi$
$t = \frac{1}{6}\pi$
$t = \frac{1}{7}\pi$
$t = \frac{1}{8}\pi$
$t = \frac{1}{9}\pi$

Theorem. At rational time $t = 2\pi\, p/q$, the solution $u^\star(t,x)$ is constant on every subinterval $2\pi\, j/q < x < 2\pi\,(j+1)/q$. At irrational time $u^\star(t,x)$ is a non-differentiable continuous function.

Lemma.

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k \, e^{\mathrm{i}\,k\,x}$$

is piecewise constant on intervals $2\pi j/q < x < 2\pi(j+1)/q$ if and only if

$$\widehat{c}_k = \widehat{c}_l, \quad k \equiv l \not\equiv 0 \mod q, \qquad \widehat{c}_k = 0, \quad 0 \neq k \equiv 0 \mod q.$$

where

$$\widehat{c}_k = \frac{2\pi k \, c_k}{\mathrm{i}\, q \,(e^{-2\,\mathrm{i}\,\pi\, k/q} - 1)} \qquad k \not\equiv 0 \mod q.$$

$\Longrightarrow$ DFT

The Fourier coefficients of the solution $u^\star(t,x)$ at rational time $t = 2\pi\, p/q$ are

$$c_k = b_k\left(2\pi\,\frac{p}{q}\right) = b_k(0)\, e^{\,\mathrm{i}\,(k\,x - 2\pi\,k^3\,p/q)},$$

where, for the step function initial data,

$$b_k(0) = \begin{cases} -\,\mathrm{i}\,/(\pi\,k), & k \text{ odd}, \\ 1/2, & k = 0, \\ 0, & 0 \neq k \;\text{ even}. \end{cases}$$

Crucial observation:

$$\text{if} \quad k \equiv l \mod q, \quad \text{then} \quad k^3 \equiv l^3 \mod q$$

and so

$$e^{\,\mathrm{i}\,(k\,x - 2\pi\,k^3\,p/q)} = e^{\,\mathrm{i}\,(l\,x - 2\pi\,l^3\,p/q)}$$

The Fundamental Solution

$$F(0,x) = \delta(x)$$

Theorem. At rational time $t = 2\pi p/q$, the fundamental solution to the initial-boundary value problem is a linear combination of finitely many periodically extended delta functions, based at $2\pi j/q$ for integers $-\frac{1}{2}q < j \leq \frac{1}{2}q$.

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Corollary. At rational time, any solution profile $u(2\pi p/q, x)$ to the periodic initial-boundary value problem depends on only finitely many values of the initial data, namely

$$u(0, x_j) = f(x_j)$$

where $x_j = x + 2\pi j/q$ for integers $-\frac{1}{2}q < j \leq \frac{1}{2}q$.

★ ★ The same quantization/fractalization phenomenon appears in any linearly dispersive equation with "integral polynomial" dispersion relation:

$$\omega(k) = \sum_{m=0}^{n} c_m k^m$$

where

$$c_m = \alpha\, n_m \qquad n_m \in \mathbb{Z}$$

Linear Free-Space Schrödinger Equation

$$\mathrm{i}\,\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Dispersion relation: $\omega = k^2$

Phase velocity: $c_p = \dfrac{\omega}{k} = -\,k$

Group velocity: $c_g = \dfrac{d\omega}{dk} = -2\,k$

Periodic Linear Schrödinger Equation

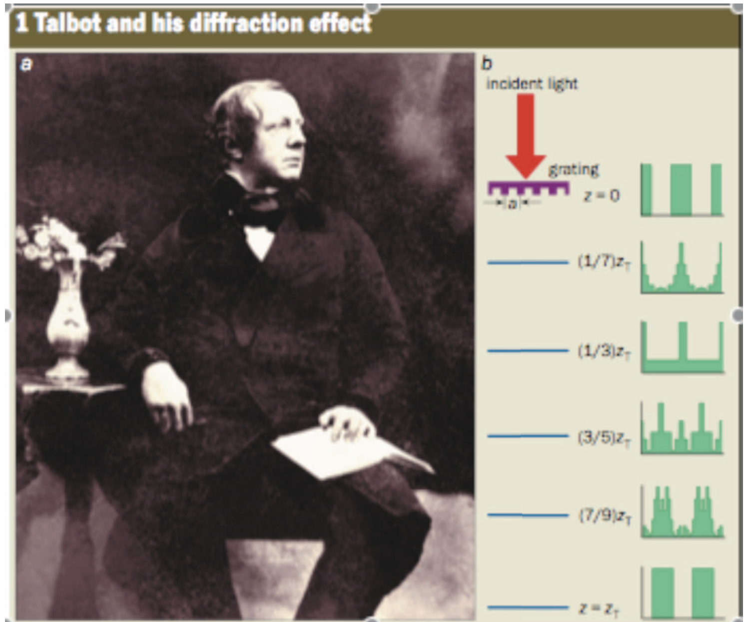
$$\mathrm{i}\,\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi)$$

- Michael Berry, et. al.
- Bernd Thaller, *Visual Quantum Mechanics*
- Oskolkov
- Kapitanski, Rodnianski

  "Does a quantum particle know the time?"
- Michael Taylor
- Fulling, Güntürk

William Henry Fox Talbot (1800–1877)



★ Talbot's 1835 image of a latticed window in Lacock Abbey

$\Longrightarrow$ oldest photographic negative in existence.

The Talbot Effect

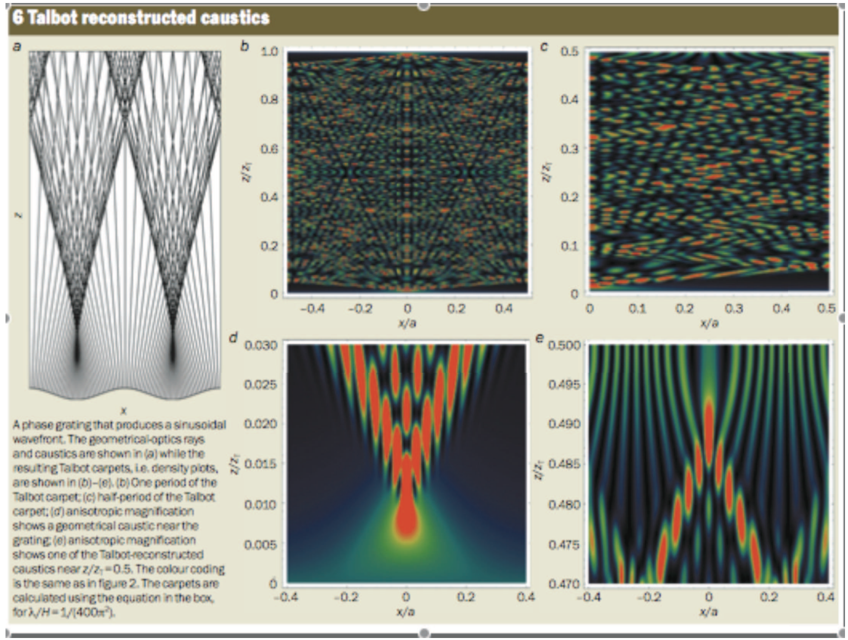
Fresnel diffraction by periodic gratings (1836)

"It was very curious to observe that though the grating was greatly out of the focus of the lens ... the appearance of the bands was perfectly distinct and well defined ... the experiments are communicated in the hope that they may prove interesting to the cultivators of optical science."

— *Fox Talbot*

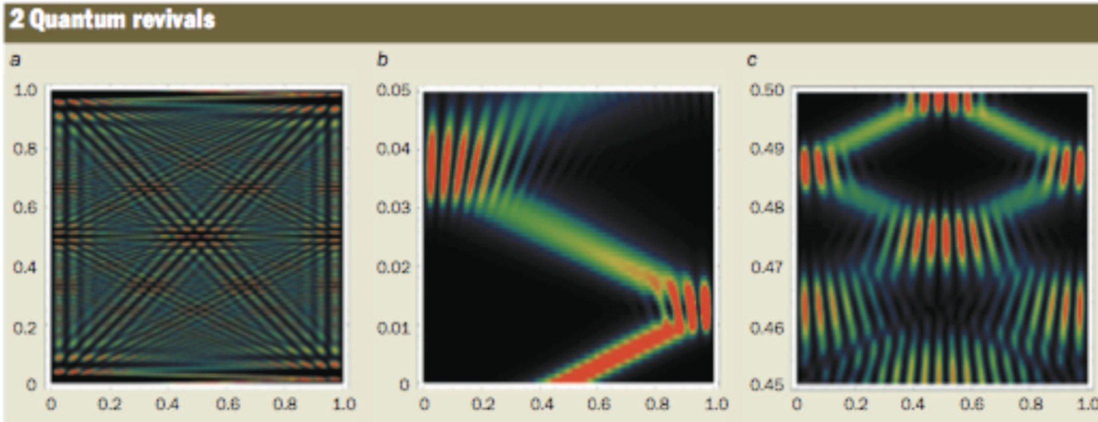
$\Longrightarrow$ Lord Rayleigh calculates the Talbot distance (1881)

The Quantized/Fractal Talbot Effect



- Optical experiments — Berry & Klein
- Diffraction of matter waves (helium atoms) — Nowak et. al.

Quantum Revival



- Electrons in potassium ions — Yeazell & Stroud
- Vibrations of bromine molecules — Vrakking, Villeneuve, Stolow

Periodic Linear Schrödinger Equation

$$\mathrm{i}\,\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi)$$

Integrated fundamental solution:

$$u(t,x) = \frac{1}{2\pi} \sum_{0 \neq k = -\infty}^{\infty} \frac{e^{\,\mathrm{i}\,(k\,x + k^2 t)}}{k}\,.$$

★ For $x/t \in \mathbb{Q}$, this is known as a Gauss (or, more generally, Weyl) sum, of importance in number theory

$\Longrightarrow$ Hardy, Littlewood, Weil, I. Vinogradov, etc.

Integrated fundamental solution:

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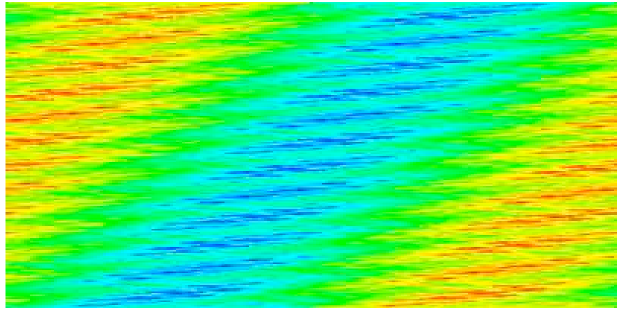
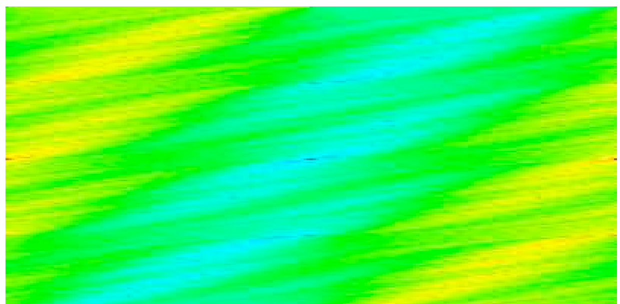
Theorem.

- The fundamental solution $\partial u/\partial x$ is a Jacobi theta function. At rational times $t = 2\pi\,p/q$, it linear combination of delta functions concentrated at rational nodes $x_j = 2\pi\,j/q$.
- At irrational times t, the integrated fundamental solution is a continuous but nowhere differentiable function. (*Claim*: The fractal dimension of its graph is $\frac{3}{2}$.)



Dispersive Carpet



Schrödinger Carpet

Periodic Linear Dispersion

$$\frac{\partial u}{\partial t} = L(D_x)\, u, \qquad u(t, x + 2\pi) = u(t, x)$$

Dispersion relation:

$$u(t,x) = e^{\,\mathrm{i}\,(k\,x - \omega\,t)} \quad \Longrightarrow \quad \omega(k) = -\,\mathrm{i}\,L(-\,\mathrm{i}\,k) \qquad \text{assumed real}$$

Riemann problem: step function initial data

$$u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

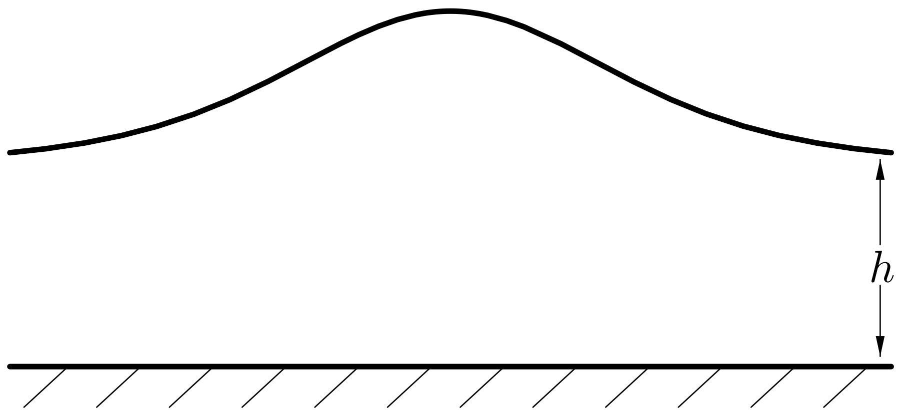
Solution:

$$u(t,x) \;\sim\; \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin[\,(2\,j+1)\,x - \omega(k)\,t\,]}{2\,j+1}\,.$$

★ ★ $\omega(-\,k) = -\,\omega(k)$ odd

Polynomial dispersion, rational t $\Longrightarrow$ Weyl exponential sums

2D Water Waves



$y = h + \eta(t, x)$ surface elevation

$\phi(t, x, y)$ velocity potential

2D Water Waves

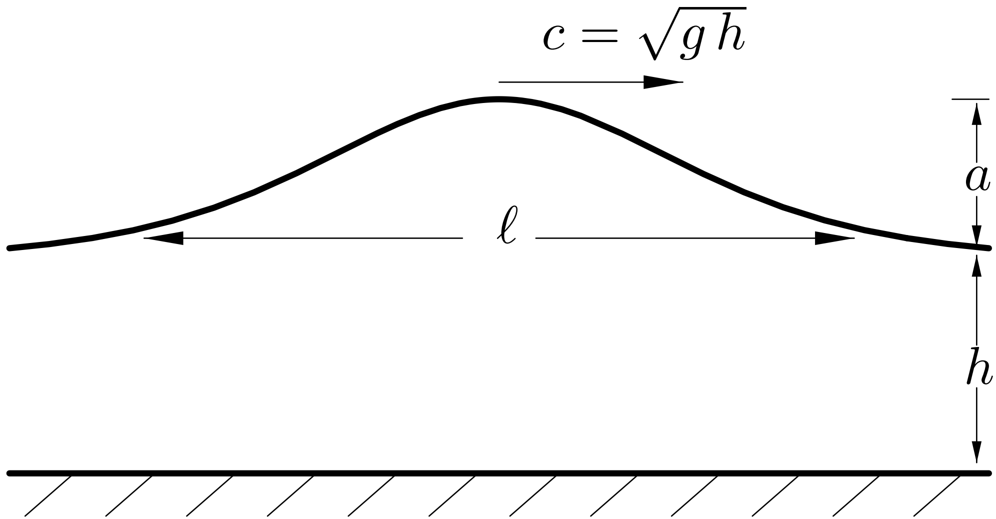
- Incompressible, irrotational fluid.
- No surface tension

$$\left.\begin{aligned} \phi_t + \tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}\phi_y^2 + g\,\eta = 0 \\ \eta_t = \phi_y - \eta_x\phi_x \end{aligned}\right\} \qquad y = h + \eta(t,x)$$

$$\phi_{xx} + \phi_{yy} = 0 \qquad\qquad 0 < y < h + \eta(t,x)$$

$$\phi_y = 0 \qquad\qquad y = 0$$

- Wave speed (maximum group velocity): $c = \sqrt{g\,h}$
- Dispersion relation: $\sqrt{g\,k\,\tanh(h\,k)} = c\,k - \frac{1}{6}c\,h^2k^3 + \cdots$



Small parameters — long waves in shallow water (KdV regime)

$$\alpha = \frac{a}{h} \qquad \beta = \frac{h^2}{\ell^2} = O(\alpha)$$

Rescale:

$$x \longmapsto \ell\, x \qquad y \longmapsto h\, y \qquad t \longmapsto \frac{\ell\, t}{c}$$

$$\eta \longmapsto a\, \eta \qquad \phi \longmapsto \frac{g\, a\, \ell\, \phi}{c} \qquad c = \sqrt{g\, h}$$

Rescaled water wave system:

$$\left.\begin{aligned} \phi_t + \frac{\alpha}{2}\phi_x^2 + \frac{\alpha}{2\beta}\phi_y^2 + \eta = 0 \\ \eta_t = \frac{1}{\beta}\phi_y - \alpha\, \eta_x\, \phi_x \end{aligned}\right\} \qquad y = 1 + \alpha\, \eta$$

$$\beta\, \phi_{xx} + \phi_{yy} = 0 \qquad 0 < y < 1 + \alpha\, \eta$$

$$\phi_y = 0 \qquad y = 0$$

Boussinesq expansion

Set

$$\psi(t,x)=\phi(t,x,0) \qquad u(t,x)=\phi_x(t,x,\theta) \qquad 0\le\theta\le 1$$

Solve Laplace equation:

$$\phi(t,x,y)=\psi(t,x)-\tfrac{1}{2}\beta^2\,y^2\,\psi_{xx}+\tfrac{1}{4!}\beta^4\,y^4\,\psi_{xxxx}+\ \cdots$$

Plug expansion into free surface conditions: To first order

$$\psi_t+\eta+\tfrac{1}{2}\alpha\,\psi_x^2-\tfrac{1}{2}\beta\,\psi_{xxt}=0$$

$$\eta_t+\psi_x+\alpha\,(\eta\psi_x)_x-\tfrac{1}{6}\beta\,\psi_{xxxx}=0$$

Bidirectional Boussinesq systems:

$$u_t + \eta_x + \alpha\, u\, u_x - \tfrac{1}{2}\beta\,(\theta^2 - 1)\, u_{xxt} = 0$$

$$\eta_t + u_x + \alpha\,(\eta\, u)_x - \tfrac{1}{6}\beta\,(3\,\theta^2 - 1)\, u_{xxx} = 0$$

★ ★ at $\theta = 1$ this system is integrable

$\Longrightarrow$ Kaup, Kupershmidt

Boussinesq equation

$$u_{tt} = u_{xx} + \tfrac{1}{2}\alpha\,(u^2)_{xx} - \tfrac{1}{6}\beta\, u_{xxxx}$$

Regularized Boussinesq equation

$$u_{tt} = u_{xx} + \tfrac{1}{2}\alpha\,(u^2)_{xx} - \tfrac{1}{6}\beta\, u_{xxtt}$$

$\Longrightarrow$ DNA dynamics (Scott)

Unidirectional waves:

$$u = \eta - \frac{1}{4}\alpha\,\eta^2 + \left(\frac{1}{3} - \frac{1}{2}\theta^2\right)\beta\,\eta_{xx} + \cdots$$

Korteweg-deVries (1895) equation:

$$\eta_t + \eta_x + \frac{3}{2}\alpha\,\eta\,\eta_x + \frac{1}{6}\beta\,\eta_{xxx} = 0$$

$\Longrightarrow$ Due to Boussinesq in 1877!

Benjamin–Bona–Mahony (BBM) equation:

$$\eta_t + \eta_x + \frac{3}{2}\alpha\,\eta\,\eta_x - \frac{1}{6}\beta\,\eta_{xxt} = 0$$

Shallow Water Dispersion Relations

Water waves	$\pm\sqrt{k\ \tanh k}$
Boussinesq system	$\pm\dfrac{k}{\sqrt{1+\frac{1}{3}k^2}}$
Boussinesq equation	$\pm k\sqrt{1+\frac{1}{3}k^2}$
Korteweg–deVries	$k-\frac{1}{6}k^3$
BBM	$\dfrac{k}{1+\frac{1}{6}k^2}$

Dispersion Asymptotics

⋆ ⋆ The qualitative behavior of the solution to the periodic problem depends crucially on the asymptotic behavior of the dispersion relation $\omega(k)$ for large wave number $k \to \pm\infty$.

$$\omega(k) \sim k^{\alpha}$$

- $\alpha = 0$ — large scale oscillations
- $0 < \alpha < 1$ — dispersive oscillations
- $\alpha = 1$ — traveling waves
- $1 < \alpha < 2$ — oscillatory becoming fractal
- $\alpha \geq 2$ — fractal/quantized

Periodic Korteweg–deVries equation

$$\frac{\partial u}{\partial t} = \alpha \, \frac{\partial^3 u}{\partial x^3} + \beta \, u \, \frac{\partial u}{\partial x} \qquad u(t, x + 2\ell) = u(t, x)$$

Zabusky–Kruskal (1965)

$$\alpha = 1, \qquad \beta = .000484, \qquad \ell = 1, \qquad u(0, x) = \cos \pi x.$$

Lax–Levermore (1983) — small dispersion

$$\alpha \longrightarrow 0, \qquad \beta = 1.$$

Gong Chen (2011)

$$\alpha = 1, \qquad \beta = 1, \qquad \ell = 2\,\pi, \qquad u(0, x) = \sigma(x).$$

Periodic Korteweg–deVries Equation

Analysis of nonsmooth initial data:

Estimates, existence, well-posedness, stability, . . .

- Kato
- Bourgain
- Kenig, Ponce, Vega
- Colliander, Keel, Staffilani, Takaoka, Tao
- Oskolkov
- D. Russell, B–Y Zhang
- Erdoğan, Tzirakis

Future Directions

- General dispersion behavior
- Other boundary conditions (Fokas)
- Higher space dimensions and other domains (tori, spheres, ...)
- Dispersive nonlinear partial differential equations
- Numerical solution techniques?
- Experimental verification in dispersive media?