

**A complete list of conservation laws for  
non-integrable compacton equations of  $K(m, m)$  type**

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## Compactons

A *compacton* is a solitary wave with compact support which preserves its shape after interacting with another compacton.

P. Rosenau, J. M. Hyman, *Compactons: Solitons with Finite Wavelength*, Phys. Rev. Lett. 70 (1993), 564–567.

## History

Compactons have first emerged as solutions of fully non-linear Korteweg-de-Vries-like equations (the  $K(m, n)$  equations):

$$u_t + D_x^3(u^n) + D_x(u^m) = 0,$$

which have first appeared in Rosenau, Hyman (1993); here  $D_x$  denotes the total  $x$ -derivative

$$D_x = \frac{\partial}{\partial x} + \sum_{j=0}^{\infty} u_{j+1} \frac{\partial}{\partial u_j},$$

$u_j$  denotes  $j$ th derivative of  $u$  with respect to  $x$ ,  $u_0 \equiv u$ ,  $m, n > 1$ ,  $t$  is the time and  $x$  is the space variable.

## Generalized $K(m, n)$ equations

Although the solitary waves have compact support only if  $n > 1$  and a compacton is a solution for a  $K(m, n)$  equation in the classical sense only for  $n \leq 3$  (Rosenau, Hyman, 1993), it is natural to study a slightly more general form of these equations (generalized  $K(m, n)$  equations):

$$u_t = aD_x^3(u^n) + bD_x(u^m), \quad (1)$$

where  $a \neq 0$  and  $b, m, n$  are arbitrary real numbers.

## Hamiltonian structure of the $K(m, m)$ equations

If  $m = n$ , these equations are easily seen to be Hamiltonian:

$$u_t = aD_x^3(u^m) + bD_x(u^m) = \mathfrak{D}\delta\mathcal{H}, \quad (2)$$

where  $\mathfrak{D} = aD_x^3 + bD_x$  is a Hamiltonian operator,  $\mathcal{H} = \int \int u^m du dx$  and  $\delta$  denotes the variational derivative of a functional with respect to  $u$ :

$$\delta \left( \int p(x, u, u_x, \dots, u_m) dx \right) \stackrel{\text{def}}{=} \frac{\delta p}{\delta u} \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} (-D_x)^j \left( \frac{\partial p}{\partial u_j} \right).$$

## Integrable or non-integrable?

The pseudo-differential operator  $\mathfrak{D}^{-1}$  is easily seen to be a formal conservation law of rank  $\infty$  for (2). This means that an infinite set of “standard” obstacles for existence of infinitely many conservation laws of increasing order for (2) vanishes, and therefore one could expect that this equation should share at least some of the properties of integrable PDEs. However, in 1996, Rosenau identified only the equations  $K(-2, -2)$  and  $K(-\frac{1}{2}, -\frac{1}{2})$  as integrable.

## Our goals

- to find out whether the generalized  $K(m, m)$  equations (1) are integrable for any other values of  $m$  than those identified by Rosenau
- to describe all symmetries and conservation laws for all non-integrable cases of generalized  $K(m, m)$  equations

## Basic definitions I

Consider an evolution equation

$$u_t = F(x, u, u_1, \dots, u_k), \quad k \geq 2. \quad (3)$$

A (smooth) function  $f$  is *local* if it depends at most on  $x, t, u$  and a finite number of  $u_j$ .

A local function  $G = G(x, t, u, u_1, \dots, u_s)$  is called a (*characteristic of*) *generalized symmetry* for (3) if

$$D_t(G) = D_F(G)$$

where  $D_t = \partial/\partial t + \sum_{j=0}^{\infty} D_x^j(F) \partial/\partial u_j$  is the total  $t$ -derivative,

and  $D_F = \sum_{j=0}^k \partial F/\partial u_j D_x^j$ .



## Basic definitions II

A *formal symmetry of order  $q$*  for (3) is a formal series of the form

$$\mathcal{L} = \sum_{j=-\infty}^1 a_j D_x^j,$$

where  $a_j$  are local functions, such that the formal series

$$D_t(\mathcal{L}) - [D_F, \mathcal{L}]$$

is of degree not greater than  $k + 1 - q$ .

The equation (3) is called *symmetry integrable* if it admits an infinite sequence of explicitly time-independent generalized symmetries of increasing order.

## The symmetry approach to integrability I

A. V. Mikhailov, A. B. Shabat, V. V. Sokolov, *The Symmetry Approach to Classification of integrable equations*, in *What is Integrability?*, V. E. Zakharov (ed.), 115–184, Springer 1991.

A. B. Shabat, A. V. Mikhailov, *Symmetries - Test of Integrability*. Important developments in soliton theory, 355–374, Springer 1993.

## The symmetry approach to integrability II

**Theorem 1** (Mikhailov, Shabat, Yamilov 1987). *An equation (3) possesses an explicitly time-independent formal symmetry of order  $N > k$  if and only if the first  $N - k$  canonical densities  $\rho_i$ ,  $i = -1, 0, 1, 2, \dots, N - k - 2$ , are densities of local conservation laws.*

*Existence of an explicitly time-independent formal symmetry of order  $q$  is a necessary condition for (3) to possess explicitly time-independent generalized symmetries with the characteristic of order  $q$ .*

## The symmetry approach to integrability III

Explicit formulas for a few first canonical densities can be found e.g. in Mikhailov, Shabat (1991). For instance, we have

$$\rho_{-1} = (\partial F / \partial u_k)^{-1/k}. \quad (4)$$

It is readily checked that for  $m \neq -2, -1/2, 0, 1$  this quantity is *not* a conserved density for (2), so by Theorem 1 equation (2) for these  $m$  has no explicitly time-independent formal symmetry and no explicitly time-independent generalized symmetry of order greater than 3.

## Symmetries of generalized $K(m, m)$ equations

**Proposition 1.** *If  $m \in \mathbb{R} \setminus \{-2, -1/2, 0, 1\}$ , then the corresponding generalized  $K(m, m)$  equation (2) has no explicitly time-independent generalized symmetries of order greater than 3; in particular, eq. (2) is not symmetry integrable.*

Using Theorem 2 from Sergyeyev, arXiv:solv-int/9902002, this result can be further extended to cover explicitly time-dependent symmetries as well:

**Proposition 2.** *If  $m \in \mathbb{R} \setminus \{-2, -1/2, 0, 1\}$ , then the corresponding generalized  $K(m, m)$  equation (2) has no generalized symmetries, including explicitly time-dependent ones, of order greater than 3.*

## Complete list of symmetries for $K(m, m)$ equations

**Theorem 2.** *For  $m \in \mathbb{R} \setminus \{-2, -1/2, 0, 1\}$  the generalized  $K(m, m)$  equation (2) has just three local generalized symmetries with the characteristics*

$$Q_1 = u_x, \quad Q_2 = u_t, \quad Q_3 = (m - 1)tu_t + u,$$

*i.e.,  $x$ - and  $t$ -translations and the scaling symmetry.*

There are no conservation laws associated (through the Hamiltonian operator  $\mathfrak{D}$ ) to the first and third symmetry.

The conserved functional associated to the second symmetry through  $\mathfrak{D}$  is the energy.

## Densities of conservation laws I

**Theorem 3.** *If  $\rho$  is a density of a local conservation law for a generalized  $K(m, m)$  equation, where  $m \in \mathbb{R} \setminus \{-2, -1/2, 0, 1\}$ , then it is, up to the addition of a trivial density, a function of  $x, t$  and  $u$  only.*

*Proof.* Let  $\rho$  be a density of a local conservation law for the  $K(m, m)$  equation,  $m \in \mathbb{R} \setminus \{-2, -1/2, 0, 1\}$ . Then  $\gamma = \frac{\delta\rho}{\delta u}$  satisfies  $D_t \left( \frac{\delta\rho}{\delta u} \right) + D_F^* \left( \frac{\delta\rho}{\delta u} \right) = 0$ , i.e.,  $\gamma$  is a cosymmetry for (2). Then  $\mathfrak{D}(\gamma) = (aD_x^3 + bD_x)(\gamma)$  is a symmetry of our equation (2). By Proposition 2,  $\text{ord } \mathfrak{D}(\gamma) \leq 3$ , therefore,  $\text{ord } \gamma \leq 0$ , i.e.,  $\gamma = \gamma(x, t, u)$ , and to it there corresponds a density  $\rho = \int \gamma du$  which also depends only on  $x, t, u$ .  $\square$

## Complete list of conservation laws

**Theorem 4.** *The only local conservation laws of the form  $D_t(\rho) = D_x(\sigma)$  for the generalized  $K(m, m)$  equation (2) with  $m \in \mathbb{R} \setminus \{-2, -1/2, 0, 1\}$ , are, modulo the addition of trivial conservation laws, just the linear combinations of the four conservation laws which for  $b \neq 0$  are given by the formulas*

$$\rho_1 = \int u^m du, \quad \sigma_1 = mau_{xx}u^{2m-1} + \frac{am(m-2)u_x^2u^{2m-2} + bu^{2m}}{2}$$

$$\rho_2 = u, \quad \sigma_2 = aD_x^2(u^m) + bu^m$$

$$\rho_3 = u \sin\left(\frac{\sqrt{b}}{\sqrt{a}}x\right), \quad \sigma_3 = aD_x^2(u^m) \sin\left(\frac{\sqrt{b}}{\sqrt{a}}x\right) - \sqrt{ab}D_x(u^m) \cos\left(\frac{\sqrt{b}}{\sqrt{a}}x\right)$$

$$\rho_4 = u \cos\left(\frac{\sqrt{b}}{\sqrt{a}}x\right), \quad \sigma_4 = aD_x^2(u^m) \cos\left(\frac{\sqrt{b}}{\sqrt{a}}x\right) + \sqrt{ab}D_x(u^m) \sin\left(\frac{\sqrt{b}}{\sqrt{a}}x\right)$$



## Complete list of conservation laws for $b = 0$

If  $b = 0$ , then the conservation law with the density  $\rho_3$  is trivial, and the densities  $\rho_2$  and  $\rho_4$  coalesce. However, there are two other conservation laws in such a case, namely

$$\rho_5 = xu, \quad \sigma_5 = aD_x^2(xu^m) - 3aD_x(u^m)$$

$$\rho_6 = x^2u, \quad \sigma_6 = aD_x^2(x^2u^m) + 6au^m - aD_x(xu^m),$$

i.e., for  $b = 0$  equation (2) with  $m \in \mathbb{R} \setminus \{-2, -1/2, 0, 1\}$  also has, up to the addition of trivial conservation laws, just four conservation laws with the densities  $\rho_1, \rho_2, \rho_5, \rho_6$  and the fluxes  $\sigma_1, \sigma_2, \sigma_5, \sigma_6$ .

## The conservation laws for $ab < 0$

If  $a$  and  $b$  have different signs, it is convenient to divide  $\rho_3$  by the imaginary unit  $i$  and use the following *real* densities and fluxes instead of the above  $\rho_3, \rho_4, \sigma_3$  and  $\sigma_4$ :

$$\tilde{\rho}_3 = cu \sinh \left( \frac{\sqrt{|b|}}{\sqrt{|a|}} x \right)$$

$$\tilde{\sigma}_3 = ca D_x^2(u^m) \sinh \left( \frac{\sqrt{|b|}}{\sqrt{|a|}} x \right) - \sqrt{|ab|} D_x(u^m) \cosh \left( \frac{\sqrt{|b|}}{\sqrt{|a|}} x \right)$$

$$\tilde{\rho}_4 = u \cosh \left( \frac{\sqrt{|b|}}{\sqrt{|a|}} x \right)$$

$$\tilde{\sigma}_4 = a D_x^2(u^m) \cosh \left( \frac{\sqrt{|b|}}{\sqrt{|a|}} x \right) - c \sqrt{|ab|} D_x(u^m) \sinh \left( \frac{\sqrt{|b|}}{\sqrt{|a|}} x \right),$$

where  $c = 1$  if  $a > 0$  and  $b < 0$ , and  $c = -1$  if  $a < 0$  and  $b > 0$ .

## Conserved functionals

The conserved functional corresponding to the first conserved density is the energy, i.e., the integral of motion associated with the invariance under the time shifts.

The remaining conserved functionals are Casimir functionals corresponding to our Hamiltonian operator  $\mathcal{D}$ , so finding a suitable physical interpretation thereof is rather unlikely.

## Conclusions

- We have proved that all the generalized  $K(m, m)$  equations where  $m \in \mathbb{R} \setminus \{-2, -1/2, 0, 1\}$  are not symmetry integrable
- We have found all generalized symmetries of the non-integrable cases of the generalized  $K(m, m)$  equations.
- We have obtained the complete list of conservation laws for the non-integrable cases of the generalized  $K(m, m)$  equations.

## The procedure of finding of all conservation laws

We consider an  $k$ -th order Hamiltonian equation  $u_t = \mathfrak{D}(\delta\mathcal{H})$

1. not all canonical densities are conserved densities  $\implies$   
symmetry non-integrability
2. only the  $N - k$  first canonical densities are conserved  
densities  $\implies N$  is the maximal order of all generalized  
time-independent symmetries
3. prove that there are no time-dependent symmetries of  
the order greater than  $N$  (**a problem!!**)
4. if  $\rho$  is a conserved density, then  $\delta\rho/\delta u$  is a cosymme-  
try and  $\mathfrak{D}(\delta\rho/\delta u)$  is a symmetry and its order is already  
known

## Outlook

It would be interesting to apply our method for proving nonexistence of higher conservation laws using existence of a Hamiltonian operator to other nonintegrable systems.

Thank you for your attention!