Elastic Sturmian spirals

P.I. Marinov, M.Ts. Hadzhilazova and I.M. Mladenov

08/6/2013

XV^{th} International Conference
"Geometry, Integrability and Quantization", Varna, Bulgaria

Overview

We consider curves, which are solutions of the Elastica equation and are generalized Sturmian Spirals as well. Analytical formulae and plots are presented.

Overview

We consider curves, which are solutions of the Elastica equation and are generalized Sturmian Spirals as well. Analytical formulae and plots are presented.

This is joint work with I. Mladenov and M. Hadzhilazova (BAS, Biophysics).

Elastica

The elastic curve minimizes for the integral of the curvature squared subject to fixed length and first order initial conditions.

Elastica

The elastic curve minimizes for the integral of the curvature squared subject to fixed length and first order initial conditions. Let $\kappa(s)$ be the curvature of a curve parametrized by arc length. We want to minimize the bending energy, which is defined simply as

Elastica

The elastic curve minimizes for the integral of the curvature squared subject to fixed length and first order initial conditions. Let $\kappa(s)$ be the curvature of a curve parametrized by arc length. We want to minimize the bending energy, which is defined simply as

$$\int_\gamma \kappa^2(s)\mathrm{d}s \tag{1}$$

Elastica

The elastic curve minimizes for the integral of the curvature squared subject to fixed length and first order initial conditions. Let $\kappa(s)$ be the curvature of a curve parametrized by arc length. We want to minimize the bending energy, which is defined simply as

$$\int_\gamma \kappa^2(s)\mathrm{d}s \tag{1}$$

with the constraints described above.

Elastica(cont.)

From the variational problem one gets the Euler-Lagrange equation for a planar curve

Elastica(cont.)

From the variational problem one gets the Euler-Lagrange equation for a planar curve

$$\kappa'' = -\frac{1}{2}\kappa^3 + \frac{\lambda}{2}\kappa \qquad (2)$$

Elastica(cont.)

From the variational problem one gets the Euler-Lagrange equation for a planar curve

$$\kappa^{''} = -\frac{1}{2}\kappa^3 + \frac{\lambda}{2}\kappa \tag{2}$$

and integrating once more one gets

$$(\kappa^{'})^2 = -\frac{1}{4}\kappa^4 + \frac{\lambda}{2}\kappa^2 + 2E \tag{3}$$

with E, λ constants. This is the classical Bernoulli-Euler Elastica.

Sturmian spirals

Sturmian spirals are defined to be planar curves that satisfy

$$\kappa = \frac{1}{r} \tag{4}$$

Sturmian spirals

Sturmian spirals are defined to be planar curves that satisfy

$$\kappa = \frac{1}{r} \tag{4}$$

The generalized ones are the ones

$$\kappa = \frac{\sigma}{r} \tag{5}$$

with $\sigma > 0$.

Direct computation

Start with the elastica equation

$$(\kappa')^2 = -\frac{1}{4}\kappa^4 + \frac{\lambda}{2}\kappa^2 + 2E \equiv P(r) \tag{6}$$

Direct computation

Start with the elastica equation

$$(\kappa')^2 = -\frac{1}{4}\kappa^4 + \frac{\lambda}{2}\kappa^2 + 2E \equiv P(r) \tag{6}$$

with the condition that

$$r = \frac{\sigma}{\kappa} \tag{7}$$

Direct computation

Start with the elastica equation

$$(\kappa^{'})^2 = -\frac{1}{4}\kappa^4 + \frac{\lambda}{2}\kappa^2 + 2E \equiv P(r) \tag{6}$$

with the condition that

$$r = \frac{\sigma}{\kappa} \tag{7}$$

so we have that

$$(r^{'})^2 = \frac{2E}{\sigma^2}r^4 + \frac{\lambda}{2}r^2 - \frac{\sigma^2}{4} \equiv Q(r). \tag{8}$$

Direct computation(cont.)

There are two cases for the coefficients of $Q(r)$, assuming $\lambda > 0$

Direct computation(cont.)

There are two cases for the coefficients of $Q(r)$, assuming $\lambda > 0$

- **Case one**: $E = \frac{a^2c^2}{8}, \qquad \lambda = \frac{a^2-c^2}{2} > 0$

Direct computation(cont.)

There are two cases for the coefficients of $Q(r)$, assuming $\lambda > 0$

- **Case one**: $E = \frac{a^2c^2}{8}, \qquad \lambda = \frac{a^2-c^2}{2} > 0$
- **Case two**: $E = -\frac{a^2c^2}{8}, \qquad \lambda = \frac{a^2+c^2}{2} > 0$

Direct computation(cont.)

We make use of the following formula for the for the solutions of the elastica (Vassilev, Djondjorov, Mladenov '08)

Direct computation(cont.)

We make use of the following formula for the for the solutions of the elastica (Vassilev, Djondjorov, Mladenov ′08)

$$x(s) = \frac{1}{\kappa^2(0) - \lambda} \int \kappa^2(s) \mathrm{d}s - \frac{\lambda s}{\kappa^2(0) - \lambda}$$
$$y(s) = \frac{-2\kappa(s)}{\kappa^2(0) - \lambda} \tag{9}$$

Direct computation(cont.)

Case one: $$r^{'} = \frac{ac}{2\sigma}\sqrt{\left(r^2 - \frac{\sigma^2}{a^2}\right)\left(r^2 + \frac{\sigma^2}{c^2}\right)}$$

$$r(s) = \frac{\sigma}{a}\frac{1}{\mathrm{cn}(\frac{\sqrt{a^2+c^2}}{2}s, k)}, \quad k = \frac{a}{\sqrt{a^2+c^2}}, \quad r > \frac{\sigma}{a} \qquad (10)$$

$$\kappa(s) = a\,\mathrm{cn}(\frac{\sqrt{a^2+c^2}}{2}s, k)$$

Direct computation(cont.)

Case one: $$r' = \frac{ac}{2\sigma}\sqrt{\left(r^2 - \frac{\sigma^2}{a^2}\right)\left(r^2 + \frac{\sigma^2}{c^2}\right)}$$

$$r(s) = \frac{\sigma}{a}\frac{1}{\mathrm{cn}(\frac{\sqrt{a^2+c^2}}{2}s, k)}, \quad k = \frac{a}{\sqrt{a^2+c^2}}, \quad r > \frac{\sigma}{a} \qquad (10)$$

$$\kappa(s) = a\,\mathrm{cn}(\frac{\sqrt{a^2+c^2}}{2}s, k)$$

Here, $\mathrm{cn}(\cdot, k)$ is Jacobi elliptic function, and k is the elliptic modulus.

Direct computation(cont.)

Case two: $$r' = \frac{ac}{2\sigma}\sqrt{\left(\frac{\sigma^2}{a^2} - r^2\right)\left(r^2 - \frac{\sigma^2}{c^2}\right)}$$

$$r(s) = \frac{\sigma}{a}\mathrm{dn}(\frac{c}{2}s, k), \quad k = \frac{\sqrt{c^2 - a^2}}{c}, \quad \frac{\sigma}{a} > r > \frac{\sigma}{c} \qquad (11)$$

$$\kappa(s) = a\mathrm{nd}(\frac{c}{2}s, k)$$

Direct computation(cont.)

Case two:

$$r' = \frac{ac}{2\sigma}\sqrt{\left(\frac{\sigma^2}{a^2} - r^2\right)\left(r^2 - \frac{\sigma^2}{c^2}\right)}$$

$$r(s) = \frac{\sigma}{a}\mathrm{dn}(\frac{c}{2}s, k), \quad k = \frac{\sqrt{c^2 - a^2}}{c}, \quad \frac{\sigma}{a} > r > \frac{\sigma}{c} \tag{11}$$

$$\kappa(s) = a\mathrm{nd}(\frac{c}{2}s, k)$$

Again, $\mathrm{dn}(\cdot, k)$, $\mathrm{nd}(\cdot, k)$ are Jacobi elliptic functions.

Direct computation(cont.)

Case two:
$$r' = \frac{ac}{2\sigma}\sqrt{\left(\frac{\sigma^2}{a^2} - r^2\right)\left(r^2 - \frac{\sigma^2}{c^2}\right)}$$

$$r(s) = \frac{\sigma}{a}\mathrm{dn}(\frac{c}{2}s, k), \quad k = \frac{\sqrt{c^2 - a^2}}{c}, \quad \frac{\sigma}{a} > r > \frac{\sigma}{c} \qquad (11)$$
$$\kappa(s) = a\mathrm{nd}(\frac{c}{2}s, k)$$

Again, $\mathrm{dn}(\cdot, k)$, $\mathrm{nd}(\cdot, k)$ are Jacobi elliptic functions.
In both cases κ is independent of σ.

Direct computation(cont.)

Here are the analytical solutions for the coordinates.

Direct computation(cont.)

Here are the analytical solutions for the coordinates.

Case one:

$$
\begin{aligned}
x(s) &= \frac{4}{\sqrt{a^2+c^2}} E(\mathrm{am}(\frac{\sqrt{a^2+c^2}}{2}s, k), k) - s \\
y(s) &= -\frac{4a}{a^2+c^2}\mathrm{cn}(\frac{\sqrt{a^2+c^2}}{2}s, k)
\end{aligned}
\tag{12}
$$

Direct computation(cont.)

Here are the analytical solutions for the coordinates.
Case one:

$$
\begin{aligned}
x(s) &= \frac{4}{\sqrt{a^2+c^2}} E\left(\mathrm{am}\left(\frac{\sqrt{a^2+c^2}}{2} s, k\right), k\right) - s \\
y(s) &= -\frac{4a}{a^2+c^2} \mathrm{cn}\left(\frac{\sqrt{a^2+c^2}}{2} s, k\right)
\end{aligned}
\tag{12}
$$

Capital E is the incomplete elliptic integral of the second kind, and $\mathrm{am}(\cdot, k)$ is the Jacobi amplitude function.

Direct computation(cont.)

Similarly for the other case.

Direct computation(cont.)

Similarly for the other case.

Case two:

$$x(s) = -\frac{4c}{c^2 - a^2} E(\mathrm{am}(\frac{c}{2}s, k), k) + \frac{4}{c}\mathrm{sn}(\frac{c}{2}s, k)\mathrm{cd}(\frac{c}{2}s, k) + \frac{a^2 + c^2}{c^2 - a^2}s$$

$$y(s) = \frac{4a}{c^2 - a^2}\mathrm{nd}(\frac{c}{2}s, k) \tag{13}$$

Direct computation(cont.)

Similarly for the other case.

Case two:

$$x(s) = -\frac{4c}{c^2 - a^2} E(\mathrm{am}(\frac{c}{2}s, k), k) + \frac{4}{c}\mathrm{sn}(\frac{c}{2}s, k)\mathrm{cd}(\frac{c}{2}s, k) + \frac{a^2 + c^2}{c^2 - a^2} s$$

$$y(s) = \frac{4a}{c^2 - a^2}\mathrm{nd}(\frac{c}{2}s, k) \tag{13}$$

Again, $\mathrm{sn}(\cdot, k)$, $\mathrm{cd}(\cdot, k)$ are Jacobi elliptic functions.

Plots

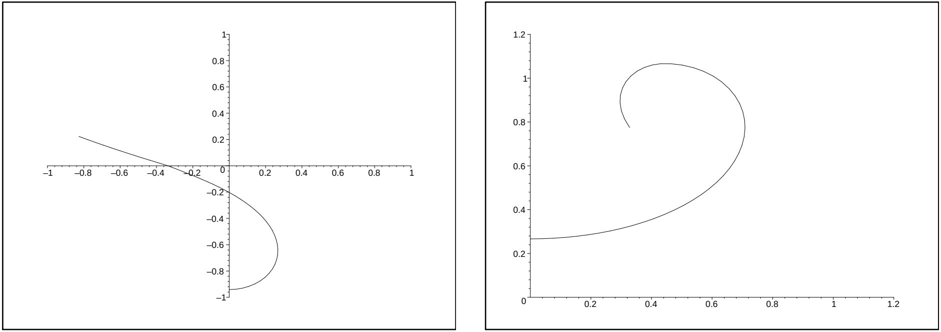

Figure: Plots of case 1 with $a = 4, c = 1$(left) and case 2 with $a = 1, c = 4$(right)

Indirect Integration

Recall the Frenet equations

$$x = \int \cos(\theta(s))\mathrm{d}s, \qquad y = \int \sin(\theta(s))\mathrm{d}s \tag{14}$$

Indirect Integration

Recall the Frenet equations

$$x = \int \cos(\theta(s))\mathrm{d}s, \qquad y = \int \sin(\theta(s))\mathrm{d}s \tag{14}$$

where θ is the angle of inclination and $\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s}$.

Indirect Integration

Recall the Frenet equations

$$x = \int \cos(\theta(s))\mathrm{d}s, \qquad y = \int \sin(\theta(s))\mathrm{d}s \tag{14}$$

where θ is the angle of inclination and $\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s}$.
Also recall $(r^{'})^2 = Q(r)$ with $r = \sigma/\kappa$.

Indirect Integration (cont.)

Rewriting we have $\mathrm{d}s = \dfrac{\mathrm{d}r}{\sqrt{Q(r)}}$.

Indirect Integration (cont.)

Rewriting we have $\mathrm{d}s = \frac{\mathrm{d}r}{\sqrt{Q(r)}}$.

Also, $\mathrm{d}\theta = \frac{\sigma \mathrm{d}s}{r}$.

Indirect Integration (cont.)

Rewriting we have $\mathrm{d}s = \frac{\mathrm{d}r}{\sqrt{Q(r)}}$.
Also, $\mathrm{d}\theta = \frac{\sigma \mathrm{d}s}{r}$.
Combining these two we have

$$\mathrm{d}\theta = \frac{\sigma \mathrm{d}r}{r\sqrt{Q(r)}} \tag{15}$$

Indirect Integration (cont.)

Rewriting we have $\mathrm{d}s = \frac{\mathrm{d}r}{\sqrt{Q(r)}}$.
Also, $\mathrm{d}\theta = \frac{\sigma \mathrm{d}s}{r}$.
Combining these two we have

$$\mathrm{d}\theta = \frac{\sigma \mathrm{d}r}{r\sqrt{Q(r)}} \tag{15}$$

Given the nature of $Q(r)$ this integrates easily. Let's try an example.

Example

Pick $Q(r) = r^4 - 1/4$, i.e. $\sigma = 1$, $\lambda = 0$ and $E = 1/2$.

Example

Pick $Q(r) = r^4 - 1/4$, i.e. $\sigma = 1$, $\lambda = 0$ and $E = 1/2$. Then

$$\mathrm{d}\theta = \frac{\sigma \mathrm{d}r}{r\sqrt{Q(r)}}$$

transforms into

$$\theta = \int \frac{2\mathrm{d}r}{r\sqrt{4r^4 - 1}} = -\cot^{-1}(\sqrt{4r^4 - 1})$$

Example

Pick $Q(r) = r^4 - 1/4$, i.e. $\sigma = 1$, $\lambda = 0$ and $E = 1/2$. Then

$$\mathrm{d}\theta = \frac{\sigma \mathrm{d}r}{r\sqrt{Q(r)}}$$

transforms into

$$\theta = \int \frac{2\mathrm{d}r}{r\sqrt{4r^4 - 1}} = -\cot^{-1}(\sqrt{4r^4 - 1})$$

or

$$2r^2 \sin(\theta) + 1 = 0 \qquad (16)$$

Example

Pick $Q(r) = r^4 - 1/4$, i.e. $\sigma = 1$, $\lambda = 0$ and $E = 1/2$.
Then

$$\mathrm{d}\theta = \frac{\sigma \mathrm{d}r}{r\sqrt{Q(r)}}$$

transforms into

$$\theta = \int \frac{2\mathrm{d}r}{r\sqrt{4r^4 - 1}} = -\cot^{-1}(\sqrt{4r^4 - 1})$$

or

$$2r^2 \sin(\theta) + 1 = 0 \tag{16}$$

Similar calculation can be done for general $Q(r)$.

Example (cont.)

Rewriting we get

$$\sin(\theta) = -\frac{1}{2r^2}$$

Example (cont.)

Rewriting we get

$$\sin(\theta) = -\frac{1}{2r^2}$$

To check that this is a spiral indeed, we differentiate

$$\cos(\theta)\kappa = \frac{1}{r^3}\frac{\mathrm{d}r}{\mathrm{d}s}$$

or

$$\frac{\sqrt{4r^4 - 1}}{2r^2}\kappa = \frac{1}{r^3}\frac{\sqrt{4r^4 - 1}}{2}$$

Example (cont.)

Rewriting we get

$$\sin(\theta) = -\frac{1}{2r^2}$$

To check that this is a spiral indeed, we differentiate

$$\cos(\theta)\kappa = \frac{1}{r^3}\frac{\mathrm{d}r}{\mathrm{d}s}$$

or

$$\frac{\sqrt{4r^4-1}}{2r^2}\kappa = \frac{1}{r^3}\frac{\sqrt{4r^4-1}}{2}$$

which shows that $\kappa = 1/r$.

Example (cont.)

Change variables $s = s(r)$ in the Frenet formulae to get

$$x = \int \cos(\theta(r))\frac{\mathrm{d}s}{\mathrm{d}r}\mathrm{d}r \qquad y = \int \sin(\theta(r))\frac{\mathrm{d}s}{\mathrm{d}r}\mathrm{d}r \tag{17}$$

Example (cont.)

Change variables $s = s(r)$ in the Frenet formulae to get

$$x = \int \cos(\theta(r))\frac{\mathrm{d}s}{\mathrm{d}r}\mathrm{d}r \qquad y = \int \sin(\theta(r))\frac{\mathrm{d}s}{\mathrm{d}r}\mathrm{d}r \tag{17}$$

therefore we get

$$\begin{aligned} x &= \int \cos(-\cot^{-1}(\sqrt{4r^4-1}))\frac{2}{\sqrt{4r^4-1}}\mathrm{d}r = \\ &\int \frac{\sqrt{4r^4-1}}{2r^2}\frac{2}{\sqrt{4r^4-1}}\mathrm{d}r = -\frac{1}{r} + c_1 = -\kappa + c_1 \\ y &= \int \sin(-\cot^{-1}(\sqrt{4r^4-1}))\frac{2}{\sqrt{4r^4-1}}\mathrm{d}r = \\ &-\int \frac{\mathrm{d}r}{r^2\sqrt{4r^4-1}} = -\frac{1}{2}\int \frac{\mathrm{d}\theta}{r} = -\frac{1}{2}\int \kappa^2(s)\mathrm{d}s \end{aligned} \tag{18}$$

with c_1 - a constant.

Example (cont.)

We see that $x(r)$ resembles the spiral property of the curve, while $y(r)$ - the elastic one.

Example (cont.)

We see that $x(r)$ resembles the spiral property of the curve, while $y(r)$ - the elastic one.
The integral for y can be reduced to standard elliptic integrals.

Example (cont.)

We see that $x(r)$ resembles the spiral property of the curve, while $y(r)$ - the elastic one.
The integral for y can be reduced to standard elliptic integrals.
This also confirms the basic structure of the solutions derived in the previous section.

Summary

We derive solutions of the classical elastica equation that are also parts of generalized Sturmian spirals. The solutions are given in closed analytical form, together with plots of certain cases. The σ in the spiral formula appear to be insignificant.

Thank you for your patience!