Kepler Problem and Formally Real Jordan Algebras II

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17^{th} International Conference on
Geometry, Integrability and Quantization
Varna, Bulgaria, June 6, 2015

Think deeply of simple things — Arnold Ross

We have learned from the last lecture that there is a family of God-given classical dynamic models (indexed by a real parameter μ) with

- configuration space $\mathbb{R}^3_*$,
- conserved angular momentum $\mathbf{L}$,
- and an additional conserved vector $\mathbf{A}$.

Moreover, the orbits of these models are either linear or conic. These models are completely integrable in the sense that

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We also learned that the orbits of these models have very attractive descriptions on the future light cone

$$\Lambda_+ := \{x \in \mathbb{R}^{1,3} | x^2 = 0, x_0 > 0\}$$

in the Minkowski space $\mathbb{R}^{1,3} := (\mathbb{R} \oplus \mathbb{R}^3, \text{Lorentz inner product})$ spanned by our ordinary three spatial dimensions and a new mysterious temporal dimension. Is this a coincidence? More precisely, one may ask this

Question: Can Kepler problem and its magnetized versions be naturally formulated on that future light cone Λ_+?

Answer: Yes, provided that we can employ the more refined Jordan algebra structure behind the Lorentz structure on that Minkowski space $\mathbb{R}^{1,3}$. Since

$$\begin{array}{ccc} \mathbb{R}^3_* & \to & \Lambda_+ \\ \mathbf{r} & \mapsto & (r, \mathbf{r}) \end{array}$$

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The Jordan algebra structure on $\mathbb{R}^{1,3}$

Write $\mathbf{x} = (x_1, x_2, x_3)$, then $x = (x_0, x_1, x_2, x_3)$. Write X for $x_0 I + \mathbf{x} \cdot \vec{\sigma}$, i.e.

$$X = \begin{bmatrix} x_0 + x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & x_0 - x_3 \end{bmatrix}.$$

Let $\mathrm{H}_2(\mathbb{C})$ is the set of all complex hermitian matrices of order two. Note that $\det X = x^2$.

- The map $x \mapsto X$ is an isometry between $\mathbb{R}^{1,3}$ and $(\mathrm{H}_2(\mathbb{C}), \det)$.
- Under the symmetrized matrix multiplication:

$$X \circ Y := \frac{1}{2}(XY + YX),$$

  $\mathrm{H}_2(\mathbb{C})$ becomes a *real commutative algebra with unit*.
- This algebra is formally real in the following sense: for A, B in $\mathrm{H}_2(\mathbb{C})$, $A^2 + B^2 = 0 \implies A = B = 0$.

Proof.

For any column vector $\vec{x}$ in $\mathbb{C}^2$, let $\vec{x}^\dagger$ be its hermitian conjugate. Then $0 = \vec{x}^\dagger(A^2 + B^2)\vec{x} = ||A\vec{x}||^2 + ||B\vec{x}||^2$, so $A\vec{x} = B\vec{x} = \vec{0}$, so $A = 0$ and $B = 0$. □

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Weak associativity

The symmetrized matrix multiplication is

- not associative.
- weakly associative in the following sense: for X, Y in $\mathrm{H}_2(\mathbb{C})$, we have
$$(X \circ Y) \circ X^2 = X \circ (Y \circ X^2).$$
Here $X^2 = X \circ X = XX$.

Proof.

$$\begin{aligned} LHS &= \frac{1}{2}(XY + YX) \circ X^2 = \frac{1}{4}[(XY + YX)X^2 + X^2(XY + YX)] \\ &= \frac{1}{4}[XYX^2 + YXX^2 + X^2XY + X^2YX] \\ &= \frac{1}{4}[XYX^2 + YX^2X + XX^2Y + X^2YX] \\ &= \frac{1}{4}[(YX^2 + X^2Y)X + X(YX^2 + X^2Y)] = X \circ \frac{1}{2}(YX^2 + X^2Y) \\ &= RHS \end{aligned}$$

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The euclidean structure on $\mathrm{H}_2(\mathbb{C})$

For any $u \in \mathrm{H}_2(\mathbb{C})$, we let L_u be the endomorphism on $\mathrm{H}_2(\mathbb{C})$ defined by $v \mapsto u \circ v$. Let $\langle\, ,\, \rangle$: $\mathrm{H}_2(\mathbb{C}) \times \mathrm{H}_2(\mathbb{C}) \to \mathbb{R}$ be defined as follows:

$$\langle u, v\rangle := \frac{1}{2}\mathrm{tr}\,(u \circ v) = \frac{1}{2}\mathrm{tr}\,(uv) = \frac{1}{4}\mathrm{tr}\,L_{u\circ v}.$$

- $\langle\, ,\, \rangle$ is an inner product on $\mathrm{H}_2(\mathbb{C})$ such that

  $$\sigma_0 := \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \sigma_1 := \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \sigma_2 := \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \sigma_3 := \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

  form an orthonormal basis. Note that $\mathrm{tr}\,\sigma_0 = 2$ and $\mathrm{tr}\,\sigma_i = 0$.
- The multiplication law for the real commutative algebra $\mathrm{H}_2(\mathbb{C})$ with unit is given by

  $$\sigma_i \circ \sigma_j = \delta_{ij}\sigma_0, \quad \sigma_0 \text{ is the unit } e.$$

- L_u is self-adjoint with respect to $\langle\, ,\, \rangle$, i.e., $\langle v, u\circ w\rangle = \langle u \circ v, w\rangle$ for any $v, w \in \mathrm{H}_2(\mathbb{C})$. Indeed,

  $$LHS = \frac{1}{2}\mathrm{tr}\,(v(uw+wu)) = \frac{1}{2}\mathrm{tr}\,(vuw+vwu) = \frac{1}{2}\mathrm{tr}\,((uv+vu)w) = RHS$$

Relevance to Kepler problem

- The future light cone Λ_+ = the set of rank one, semi-positive elements in $\mathrm{H}_2(\mathbb{C})$. Indeed, if the rank of X is less than two, then $\det X = 0$, also, if $X \neq 0$ is semi-positive, then $\operatorname{tr} X > 0$. So $x^2 = 0$ and $x_0 > 0$.
- For the Kepler problem, the potential term is

$$-\frac{1}{\langle e, x\rangle}.$$

- The Kinetic term for the the Kepler problem (or rather the Riemannian metric on Λ_+), angular momentum, and Lenz vector can all be naturally expressed in terms of Jordan algebra structure as well.

For details, we have to study Formally Real Jordan algebras.

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Formally real Jordan algebras

Jordan algebras are the unfavored cousins of Lie algebras, and Formally real Jordan algebras are the unfavored cousins of compact real Lie algebras. Having $\mathrm{H}_2(\mathbb{C})$ in mind, we have

Definition (P. Jordan, 1933)

A finite dimensional **Formally real Jordan algebra** is a finite dimensional real algebra V with unit e such that, for any elements a, b in V, we have
1) $ab = ba$ (symmetry),
2) $a(ba^2) = (ab)a^2$ (weakly associative),
3) $a^2 + b^2 = 0 \implies a = b = 0$ (formally real).

The simplest example is $\mathbb{R}$, the other example is $\mathrm{H}_2(\mathbb{C})$. We use L_a: $V \to V$ to denote the multiplication by a. Then 2) says that $[L_a, L_{a^2}] = 0$ (Jordan Identity) and 3) can be replaced by

3´) The "Killing form" $\langle a, b\rangle = \frac{1}{\dim V}\mathrm{tr}\, L_{ab}$ is positive definite. Note that $\langle b, ac\rangle = \langle ab, c\rangle$. Formally real Jordan algebras are also called Euclidean Jordan algebras.

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The classification theorem

Theorem (Jordan, von Neumann and Wigner, 1934)

Euclidean Jordan algebras are semi-simple, and the simple ones consist of four infinity families and one exceptional:

$\mathbb{R}$.

$\Gamma(n) := \mathbb{R} \oplus \mathbb{R}^n$, $n \geq 2$.

$\mathrm{H}_n(\mathbb{R})$, $n \geq 3$.

$\mathrm{H}_n(\mathbb{C})$, $n \geq 3$.

$\mathrm{H}_n(\mathbb{H})$, $n \geq 3$.

$\mathrm{H}_3(\mathbb{O})$.

Remark.

- $\Gamma(0) \cong \mathbb{R}$, $\Gamma(1) \cong \mathbb{R} \oplus \mathbb{R}$, $\Gamma(2) \cong \mathrm{H}_2(\mathbb{R})$, $\Gamma(3) \cong \mathrm{H}_2(\mathbb{C})$, $\Gamma(5) \cong \mathrm{H}_2(\mathbb{H})$, $\Gamma(9) \cong \mathrm{H}_2(\mathbb{O})$.
- Each but the exceptional one is associated with an associative algebra.
- $\mathbb{R}$, $\Gamma(3)$, and $\mathrm{H}_3(\mathbb{O})$ are somewhat special.

The structure algebra

For a, b in V, we let

$$S_{ab} := [L_a, L_b] + L_{ab}, \quad \{abc\} := S_{ab}(c)$$

and $\mathfrak{str}$ be the span of $\{S_{ab} \mid a, b \in V\}$ over $\mathbb{R}$. Since

$$[S_{ab}, S_{cd}] = S_{\{abc\}d} - S_{c\{bad\}},$$

$\mathfrak{str}$ becomes a real Lie algebra — the **structure algebra** of V. For example, (1) $\mathfrak{str} \cong \mathbb{R}$ for $V = \mathbb{R}$, (2) $\mathfrak{str} \cong \mathfrak{so}(1,3) \oplus \mathbb{R}$ for $V = \Gamma(3)$.

This Lie algebra is not simple, actually not even semi-simple, because it has a non-trivial central element: S_{ee}. Note that $L_u = S_{ue}$.

The good news is that this algebra can be extended to a simple real Lie algebra provided that V is a simple Euclidean Jordan algebra. From hereon V is assumed to be a simple Euclidean Jordan algebra .

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$$[S_{ab}, S_{cd}] = S_{\{abc\}d} - S_{c\{bad\}},$$

$\mathfrak{str}$ becomes a real Lie algebra — the **structure algebra** of V. For example, (1) $\mathfrak{str} \cong \mathbb{R}$ for $V = \mathbb{R}$, (2) $\mathfrak{str} \cong \mathfrak{so}(1, 3) \oplus \mathbb{R}$ for $V = \Gamma(3)$.

This Lie algebra is not simple, actually not even semi-simple, because it has a non-trivial central element: S_{ee}. Note that $L_u = S_{ue}$.

The good news is that this algebra can be extended to a simple real Lie algebra provided that V is a simple Euclidean Jordan algebra. From hereon V is assumed to be a simple Euclidean Jordan algebra .

The conformal algebra

Write $z \in V$ as X_z and $\langle w, \rangle \in V^*$ as Y_w.

Definition (J. Tits, M. Koecher, I.L. Kantor, 1960's)

The **conformal algebra** $\mathfrak{co}$ is a Lie algebra whose underlying real vector space is $V \oplus \mathfrak{str} \oplus V^*$, and the commutation relations are

$$\boxed{\begin{gathered}[X_u, X_v] = 0, \quad [Y_u, Y_v] = 0, \quad [X_u, Y_v] = -2S_{uv}, \\ [S_{uv}, X_z] = X_{\{uvz\}}, \quad [S_{uv}, Y_z] = -Y_{\{vuz\}}, \\ [S_{uv}, S_{zw}] = S_{\{uvz\}w} - S_{z\{vuw\}}\end{gathered}} \tag{1}$$

for u, v, z, w in V.

When $V = \Gamma(3)$, $\mathfrak{str} = \mathfrak{so}(3,1) \oplus \mathbb{R}$, $\mathfrak{co} = \mathfrak{so}(4,2)$. When $V = \mathbb{R}$, $\mathfrak{str} = \mathbb{R}$, $\mathfrak{co} = \mathfrak{sl}(2, \mathbb{R})$. In general, $\mathfrak{co}$ is the Lie algebra of the bi holomorphic automorphism group of the complex domain $V \oplus \mathrm{i} V_+ \subset V \otimes_{\mathbb{R}} \mathbb{C}$.

The universal Kepler problem [G. Meng, "The Universal Kepler Problems", JGSP 36 (2014) 47-57]

Let $\mathcal{TKK}$ be the complexified universal enveloping algebra for the conformal algebra, but with Y_e being formally inverted.

Definition

The **universal angular momentum** is

$$\begin{aligned} L: V\times V &\to \mathcal{TKK} \\ (u,v) &\mapsto L_{u,v} := [L_u, L_v] \end{aligned} \qquad (2)$$

The **universal Hamiltonian** is

$$H := \frac{1}{2} Y_e^{-1} X_e - (\mathrm{i} Y_e)^{-1} \qquad (3)$$

The **universal Lenz vector** is

$$\begin{aligned} A: V &\to \mathcal{TKK} \\ u &\mapsto A_u := (\mathrm{i} Y_e)^{-1}[L_u, (\mathrm{i} Y_e)^2 H] \end{aligned} \qquad (4)$$

Universal Lenz algebra

Using the commutation relation for the conformal algebra, one can prove the following

Theorem

For u, v, z and w in V,

$$\boxed{\begin{array}{rcl} [L_{u,v}, H] & = & 0\ , \\ [A_u, H] & = & 0\ , \\ [L_{u,v}, L_{z,w}] & = & L_{[L_u,L_v]z,w} + L_{z,[L_u,L_v]w}\ , \\ [L_{u,v}, A_z] & = & A_{[L_u,L_v]z}\ , \\ [A_u, A_v] & = & -2HL_{u,v}\ . \end{array}} \tag{5}$$

A concrete realization of the conformal algebra

$\Downarrow$

a concrete model of the Kepler type

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To be continued

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