On a problem of David A. Singer about prescribing curvature for curves

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Fundamental Theorem for plane curves

THEOREM

Prescribe $\kappa = \kappa(s)$ (continuous):
$\theta(s) = \int \kappa(s)\,ds, \quad x(s) = \int \cos\theta(s)\,ds, \quad y(s) = \int \sin\theta(s)\,ds$
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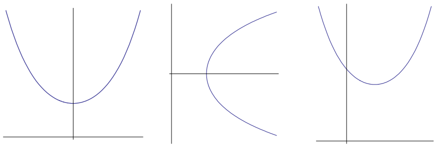
Example (Catenary)

$\kappa(s) = \frac{1}{1+s^2} \Rightarrow \theta(s) = \arctan s$

$\quad x(s) = \log\left(s + \sqrt{s^2+1}\right),\ y(s) = \sqrt{1+s^2} \leftrightarrow y = \cosh x,\ x \in \mathbb{R}$



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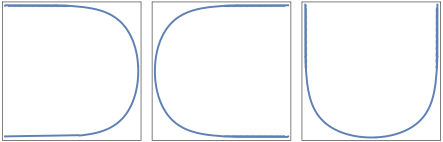
Example (Grim reaper)

$\kappa(s) = \operatorname{sech} s \Rightarrow \theta(s) = 2 \arctan e^s$

$x(s) = -\log\cosh s,\ y(s) = 2\arctan e^s \leftrightarrow x = \log\sin y,\ 0 < y < \pi$



Singer's Problem

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Can a plane curve be determined if its curvature is given in terms of its position?

$$\kappa = \kappa(x, y), \qquad \frac{x'(t)y''(t) - y'(t)x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}} = \kappa(x(t), y(t))$$

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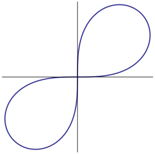
Bernoulli lemniscate: $r^2 = 3 \sin 2\theta$



Euler's elastic curves

Elastica under *tension* $\sigma \in \mathbb{R}$:
Critical points of $\int (\kappa^2 + \sigma) ds$: $2\ddot{\kappa} + \kappa^3 - \sigma\kappa = 0$

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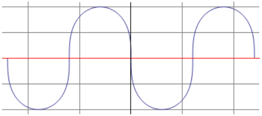

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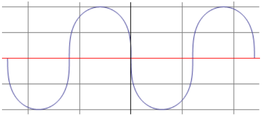

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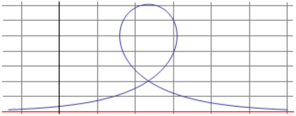

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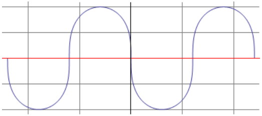

- $c = -1$, *borderline*:

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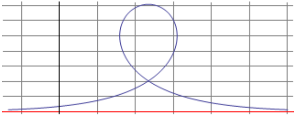

- $c < -1$, *orbitlike*:

$\kappa(s) = k_0 \ \mathrm{dn}\left(\frac{k_0 s}{2}, p\right)$,
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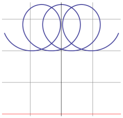

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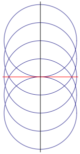
Example (Circles)

$$\kappa \equiv \kappa_0 > 0, \quad \mathcal{K}(y) = \kappa_0 y + c$$
$$s = \int \frac{dy}{\sqrt{1-(\kappa_0 y + c)^2}} = \frac{\arcsin(\kappa_0 y + c)}{\kappa_0}$$
$$y(s) = \frac{\sin(\kappa_0 s) - c}{\kappa_0}, \quad x(s) = \frac{\cos(\kappa_0 s)}{\kappa_0}$$



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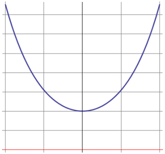

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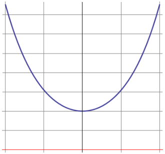
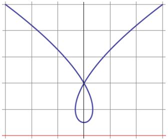


- $\mathcal{K}(y) = 1 - 1/y$

$$9x^2 = (y-2)^2(2y-1)$$



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- <u>$0 < \lambda < 1$</u>

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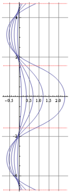

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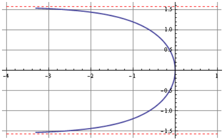

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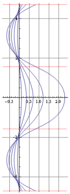

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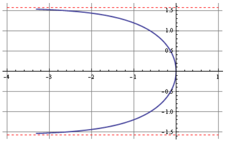

- $\underline{\lambda > 1}$

$$y(s) = \arcsin\left(\frac{\mathrm{sn}(\lambda s, \frac{1}{\lambda})}{\lambda}\right)$$
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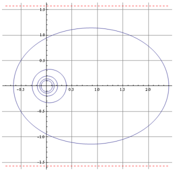

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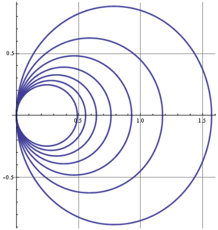

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$$y(s) = \log\left(\frac{\cosh(\sqrt{1-c^2}s)-c}{1-c^2}\right) \quad \kappa(s) = \frac{1-c^2}{\cosh(\sqrt{1-c^2}s)-c}, \ s \in \mathbb{R}$$

$$x(s) = 2\arctan\left(\frac{e^{\sqrt{1-c^2}s}-c}{\sqrt{1-c^2}}\right) - cs$$

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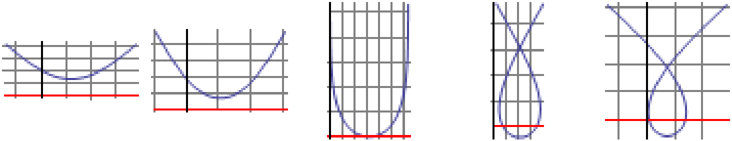
$$y(s) = \log\left(\frac{\cosh(\sqrt{1-c^2}s)-c}{1-c^2}\right) \quad \kappa(s) = \frac{1-c^2}{\cosh(\sqrt{1-c^2}s)-c}, \; s \in \mathbb{R}$$

$$x(s) = 2\arctan\left(\frac{e^{\sqrt{1-c^2}s}-c}{\sqrt{1-c^2}}\right) - cs$$



Plane curves such that $\boxed{\kappa(y) = \lambda / e^y \ (\lambda = 1)}$

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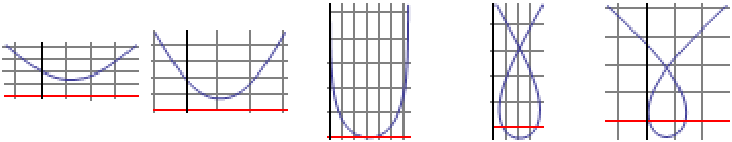

- ▶ $c = 0$: $\mathcal{K}(y) = -e^{-y}$ Grim-reaper $y = -\log \cos x, \ -\pi/2 < x < \pi/2$

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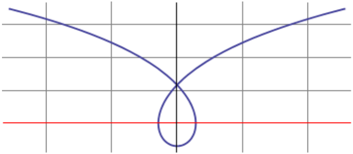
Alysoid



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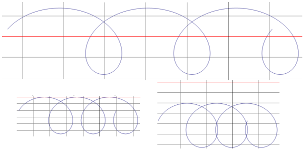

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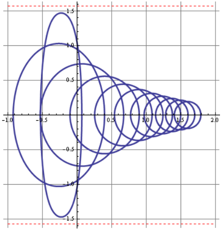

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- <u>$0 < \lambda < 1$</u>

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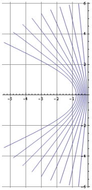

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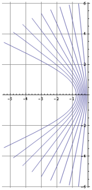

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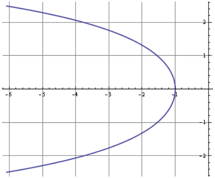
Catenary:
$x = -\cosh y, \ y \in \mathbb{R}$



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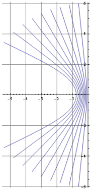

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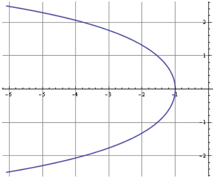

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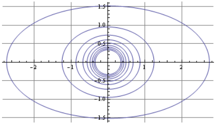
$$x(s) = \frac{\lambda}{\sqrt{\lambda^2-1}} \arcsin\left(\frac{\cos(\sqrt{\lambda^2-1}s)}{\lambda}\right)$$



Uniqueness results

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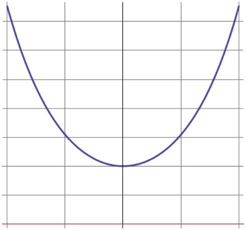
The **catenary** $y = \cosh x$, $x \in \mathbb{R}$,
is the only plane curve (up to x-translations)
with geometric linear momentum $\mathcal{K}(y) = -1/y$
(and curvature $\boxed{\kappa(y) = 1/y^2}$).



The **catenary** $x = -\cosh y$, $y \in \mathbb{R}$,
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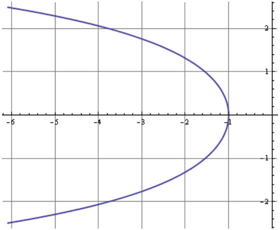

Uniqueness results

The **grim-reaper** $y = -\log \sin x$, $0 < x < \pi$,
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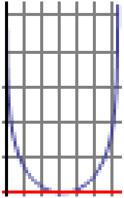

The **grim-reaper** $x = \log \cos y$, $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$,
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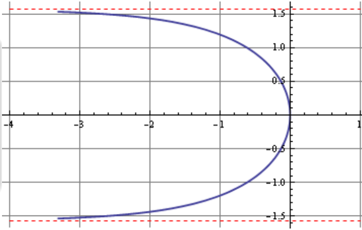

Index

Curves with curvature depending on distance from a point

P.A. Djondjorov, M.T. Hadzhilazova, P.I. Marinov, I.M. Mladenov, V.M. Vassilev, . . .

[I. Castro, I. Castro-Infantes and J. Castro-Infantes: *New plane curves with curvature depending on distance from the origin*. Mediterr. J. Math. **14** (2017), 108:1–19.]

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Prescribe $\kappa = \kappa(r)$ such that $r\kappa(r)$ continuous.
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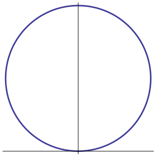
Example (Circles)

$\kappa \equiv k_0 > 0, \quad \mathcal{K}(r) = k_0 r^2/2 + c$

$s = \int \frac{r\,dr}{\sqrt{r^2 - (k_0 r^2/2 + c)^2}} \overset{(c=0)}{=} (2/k_0)\arcsin(k_0 r/2)$

$r(s) = (2/k_0)\sin(k_0 s/2), \ \theta(s) = k_0 s/2$



Plane curves such that $\boxed{\kappa(r) = 1/r}$

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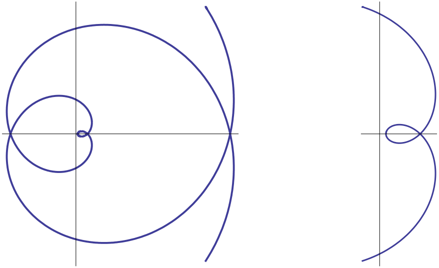
Sturm or Norwich spiral



Plane curves such that $\boxed{\kappa(r) = \lambda\, r^{n-1}\ (\lambda > 0, n \neq -1, 0)}$

$\mathcal{K}(r) = \frac{\lambda}{n+1} r^{n+1}$

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$$d\theta = \frac{\frac{\lambda}{n+1}\, r^{n-1}}{\sqrt{1 - \left(\frac{\lambda}{n+1}\right)^2 r^{2n}}} dr$$

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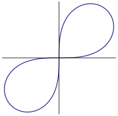
- $n = 2$: *Bernoulli lemniscate* $r^2 = \frac{3}{\lambda} \sin 2\theta$



Plane curves such that $\kappa(r) = \lambda\, r^{n-1}$ $(\lambda > 0, n \neq -1, 0)$

$\mathcal{K}(r) = \frac{\lambda}{n+1} r^{n+1}$

[Djondjorov, Vassilev and Mladenov, 2009]
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$$d\theta = \frac{\frac{\lambda}{n+1}\, r^{n-1}}{\sqrt{1 - \left(\frac{\lambda}{n+1}\right)^2 r^{2n}}}\, dr$$

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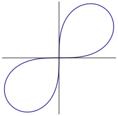
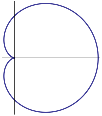


- $n = 1/2$: *Cardioid* $r = \frac{9}{4\lambda^2} \sin^2 \frac{\theta}{2}$



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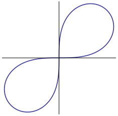

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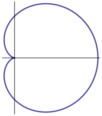
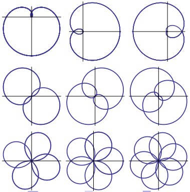


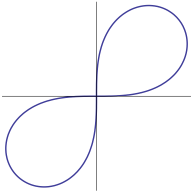
- $n \in \mathbb{Q}$:

Algebraic curves
$n = 1/3, 1/4, 1/6$
$n = 2/3, 2/5, 2/7$
$n = 4/3, 5/4, 6/5$

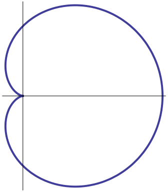


Uniqueness results for plane curves

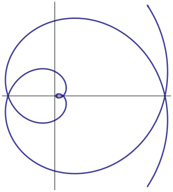
The **Bernoulli lemniscate** $r^2 = 3\sin 2\theta$
is the only plane curve (up to rotations)
with geometric angular momentum $\mathcal{K}(r) = r^3/3$
(and curvature $\boxed{\kappa(r) = r}$).



The **cardioid** $r = \frac{1}{2}(1 + \cos\theta)$
is the only plane curve (up to rotations)
with geometric angular momentum $\mathcal{K}(r) = r\sqrt{r}$
(and curvature $\boxed{\kappa(r) = \frac{3}{2\sqrt{r}}}$).



The **Norwich spiral** is the only (non circular)
plane curve (up to rotations) with curvature
$\boxed{\kappa(r) = 1/r}$.



Plane curves such that $\boxed{\kappa(r) = \frac{\lambda}{r^3} + 3\mu r \ (\lambda \in \mathbb{R}, \mu > 0)}$

$\mathcal{K}(r) = \mu r^3 - \lambda / r$

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[Mladenov, Hadzhilazova, Djondjorov and Vassilev, 2011]

$$d\theta = \frac{\mu r^4 - \lambda}{r\sqrt{r^4 - (\mu r^4 - \lambda)^2}} dr$$

- $1 + 4\lambda\mu > 0$:

$\exists \ a, b \neq 0 \ / \ \mu = \frac{1}{2b^2}, \ \lambda = \frac{a^4 - b^4}{2b^2}$

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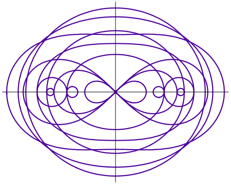
Cassini ovals $r^4 - 2a^2 r^2 \cos 2\theta + a^4 = b^4$



$a \in \{1, 2, 3\}, b \in \{1, 2, 3, 4\}$

Plane curves such that $\boxed{\kappa(r) = 2\lambda + \mu/r \ (\lambda = 1, \mu \neq 0)}$

$\mathcal{K}(r) = r^2 + \mu r$, $\mu < 1$

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$r(s) = \cos s - \mu$, $\theta(s) = s + \mu \int \frac{ds}{\cos s - \mu}$ $\qquad \kappa(s) = \dfrac{2\cos s - \mu}{\cos s - \mu}$

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  $(s_\gamma \equiv \sin\gamma, \ c_\gamma \equiv \cos\gamma, \ t_\gamma \equiv \tan\gamma)$

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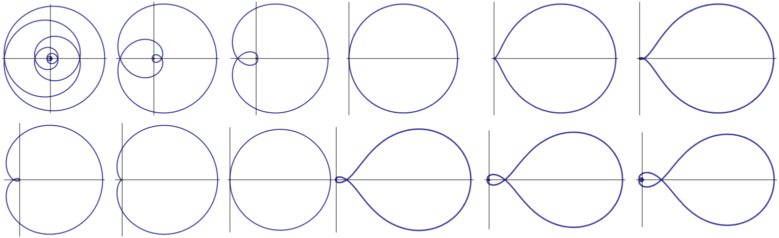
  $(s_\gamma \equiv \sin\gamma, \ c_\gamma \equiv \cos\gamma, \ t_\gamma \equiv \tan\gamma)$

$$r_\gamma(s) = \cos s - c_\gamma, \ \theta_\gamma(s) = s + \frac{2}{t_\gamma} \operatorname{arctanh}\left(\frac{s_\gamma}{1 - c_\gamma} \tan\frac{s}{2}\right)$$



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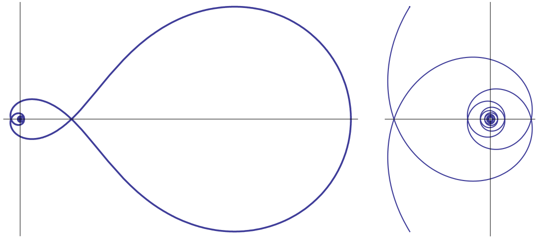
- $\underline{\mu = -1:}$

Inverse Norwich spiral

$r(s) = \cos s + 1, \ \theta(s) = s - \tan \frac{s}{2}$



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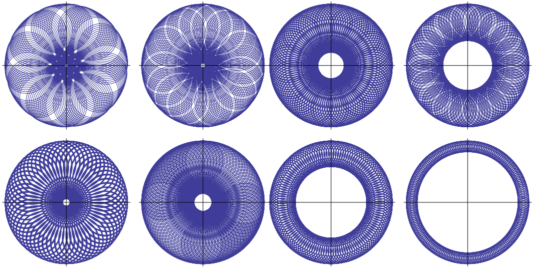
$\mathcal{K}(r) = r^2 + \mu r, \ \mu < 1$

$r(s) = \cos s - \mu, \ \theta(s) = s + \mu \int \frac{ds}{\cos s - \mu}$ $\qquad \kappa(s) = \dfrac{2\cos s - \mu}{\cos s - \mu}$

- $\underline{\mu < -1}$: $\mu = -\cosh\delta, \ \delta > 0$
  $(s_\delta \equiv \sinh\delta, \ c_\delta \equiv \cosh\delta, \ t_\delta \equiv \tanh\delta)$

$$r_\delta(s) = \cos s + c_\delta, \ \theta_\delta(s) = s - \frac{2}{t_\delta} \arctan\left(\frac{s_\delta}{1 + c_\delta} \tan\frac{s}{2}\right)$$



Plane curves such that $\boxed{\kappa(r) = \lambda/\sqrt{r^2+1}\ (0 < \lambda < 1)}$

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$\lambda = \sin\beta\ ,\ \beta \in (0, \pi/2)\ (s_\beta \equiv \sin\beta,\ c_\beta \equiv \cos\beta,\ t_\beta \equiv \tan\beta)$

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$\lambda = \sin\beta$, $\beta \in (0, \pi/2)$ $(s_\beta \equiv \sin\beta,\ c_\beta \equiv \cos\beta,\ t_\beta \equiv \tan\beta)$

$$r_\beta(t)^2 = \frac{\cosh^2(c_\beta t)}{c_\beta^2} - 1,\ \theta_\beta(t) = s_\beta t + \arctan\left(\frac{\tanh(c_\beta t)}{t_\beta}\right)$$

$$(ds = \sqrt{r_\beta^2 + 1}dt,\ \ t = \sinh(c_\beta t)/c_\beta^2)$$

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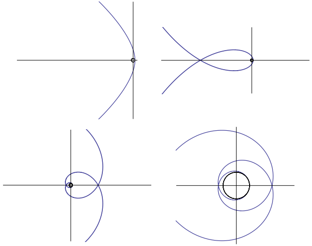
$$\kappa_\beta(s) = \frac{\sin\beta\cos\beta}{\sqrt{1+\cos^4\beta\, s^2}},\ \beta \in (0, \pi/2)$$



Plane curves such that $\boxed{\kappa(r) = \lambda/\sqrt{r^2-1}\ (\lambda > 0)}$ $\mathcal{K}(r) = \lambda\sqrt{r^2-1}$

- $\underline{0 < \lambda < 1}$: $\lambda = \sin\alpha$, $\alpha \in (0, \pi/2)$
  $(s_\alpha \equiv \sin\alpha,\ c_\alpha \equiv \cos\alpha,\ t_\alpha \equiv \tan\alpha)$

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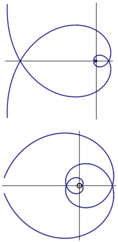

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Anti-clothoid

$$r(s) = \sqrt{1 + 2s},\ \theta(s) = \sqrt{2s} - \arctan\sqrt{2s}$$

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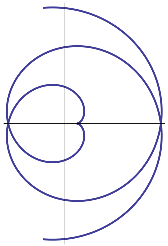

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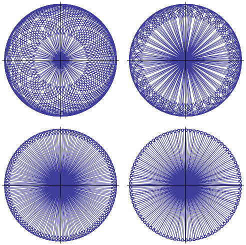

Index

Spherical version of Singer's Problem

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Can a spherical curve be determined
if its (geodesic) curvature is given in terms of its position?

$$\begin{vmatrix} x(s) & y(s) & z(s) \\ \dot{x}(s) & \dot{y}(s) & \dot{z}(s) \\ \ddot{x}(s) & \ddot{y}(s) & \ddot{z}(s) \end{vmatrix} = \kappa(x(s), y(s), z(s))$$

$$x(s)^2 + y(s)^2 + z(s)^2 = 1, \; \dot{x}(s)^2 + \dot{y}(s)^2 + \dot{z}(s)^2 = 1$$

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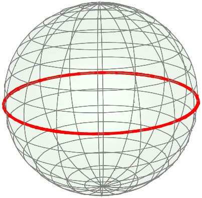
[I. Castro, I. Castro-Infantes and J. Castro-Infantes. *Spherical curves whose curvature depends on distance to a great circle*. Preprint.]

$\kappa(x, y, z) = \kappa(z)$, $z = \sin\varphi$, φ latitude



Spherical curves whose curvature depends on distance to a geodesic (or from a point)

Spherical curves whose curvature depends on distance to a geodesic (or from a point)

Theorem

Prescribe $\kappa = \kappa(z)$ continuous.
The problem of determining a spherical curve
$\xi(s) = (x(s), y(s), z(s))$ *-s arc parameter- whose curvature is* $\kappa(z)$*,*
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3. $x(s) = \cos\varphi(s)\cos\lambda(s),\ \ y(s) = \cos\varphi(s)\sin\lambda(s),$
   $\varphi(s) = \arcsin z(s),\ \lambda(s) = \int \frac{\mathcal{K}(z(s))}{z(s)^2 - 1} ds$

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3. $x(s) = \cos\varphi(s)\cos\lambda(s), \; y(s) = \cos\varphi(s)\sin\lambda(s),$
   $\varphi(s) = \arcsin z(s), \; \lambda(s) = \int \frac{\mathcal{K}(z(s))}{z(s)^2 - 1} ds$

- ξ *is uniquely determined (up to rotations around the z-axis) by* $\mathcal{K}(z)$

Examples

Example (Great circles)

$\kappa \equiv 0$: $\int \kappa(z)\,dz = c$,

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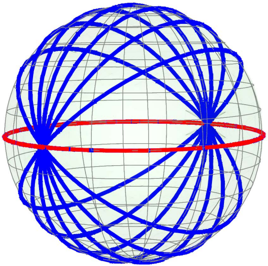

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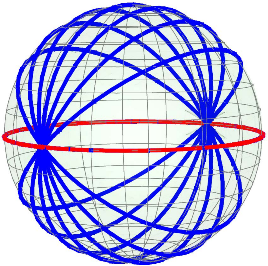


Example (Small circles)

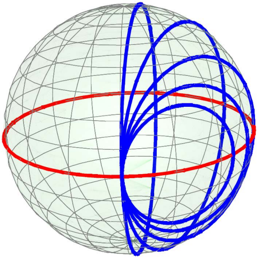
$\kappa \equiv k_0 \geq 0:$ $\int \kappa(z)\,dz = k_0 z + c$

$z(s) =$

$\frac{1}{1+k_0^2}\left(\sqrt{1-c^2+k_0^2}\,\sin(\sqrt{1+k_0^2}\,s) - c\,k_0\right)$,

$|c| < \sqrt{1+k_0^2}$.

$c = 0$: $\mathbb{S}^2 \cap \{y = \frac{k_0}{\sqrt{1+k_0^2}}\}$, $\mathcal{K}(z) = k_0 z$



Spherical elasticae: characterization and generalization

Elasticae under tension $\sigma \in \mathbb{R}$:
critical points of $\mathcal{F}_\sigma(\xi) := \int_\xi (\kappa^2 + \sigma) ds$ ($\sigma = 0$ *free* elasticae)

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(ii) *Conversely, ξ critical point of*
$\mathcal{F}^\lambda_\sigma(\xi) := \int_\xi ((\kappa + \lambda)^2 + \sigma) ds$, $\lambda, \sigma \in \mathbb{R}$
$\Rightarrow \exists a \neq 0, b \in \mathbb{R}$: $\kappa(z) = 2az + b$

Spherical *borderline* elastic curves.

- $a > 1/2$, $b = 0$, $c = 1$: $\boxed{\kappa(z) = 2az}$, $a > 0$, $\mathcal{K}(z) = az^2 + 1$

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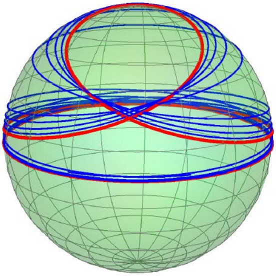

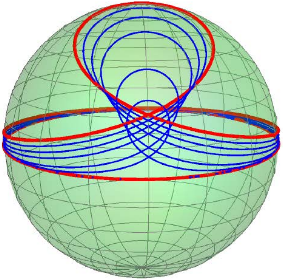

Seiffert's spherical elastic *spirals*

$r = \text{sn}(s, k),\ \theta = k\,s,\ z = \text{cn}(s, k),\ (k > 0)$ [Erdös, 2000]

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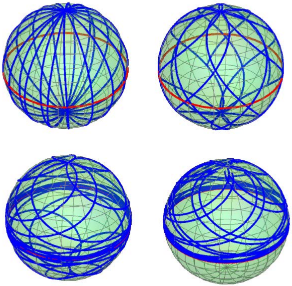


New spherical curves I:

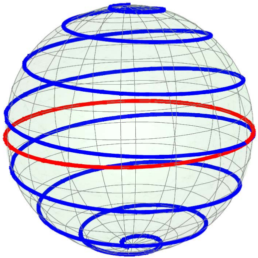
$$\boxed{\kappa(z) = \frac{z}{\sqrt{a-z^2}}}, \quad 0<a=\sin^2\alpha<1 \ (0<\alpha<\pi/2)$$

$$\mathcal{K}(z) = -\sqrt{\sin^2\alpha - z^2}$$

$$\varphi(s) = \arcsin(\cos\alpha\, s)$$

$$\kappa(s) = \frac{c_\alpha s}{\sqrt{s_\alpha^2 - c_\alpha^2 s^2}}, \quad |s| < \tan\alpha$$

$$\lambda(s) = \frac{1}{c_\alpha}\arctan\left(\frac{c_\alpha s}{\sqrt{s_\alpha^2 - c_\alpha^2 s^2}}\right) - \frac{1}{2}\arctan\left(\frac{c_\alpha s + s_\alpha^2}{c_\alpha \sqrt{s_\alpha^2 - c_\alpha^2 s^2}}\right) - \frac{1}{2}\arctan\left(\frac{c_\alpha s - s_\alpha^2}{c_\alpha \sqrt{s_\alpha^2 - c_\alpha^2 s^2}}\right)$$



New spherical curves II:

$$\boxed{\kappa(z) = \frac{az}{\sqrt{1-az^2}}}, \quad a = \cosh^2\delta > 1, \ (\delta > 0)$$

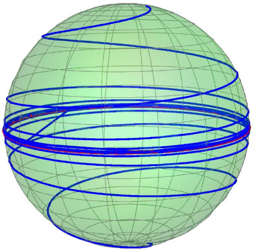
$$\mathcal{K}(z) = -\sqrt{1-\cosh^2\delta z^2}$$

$$\varphi(s) = \arcsin(e^{\sinh\delta\, s})$$

$$\kappa(s) = \frac{\cosh^2\delta\, e^{\sinh\delta\, s}}{\sqrt{1-\cosh^2\delta\, e^{2\sinh\delta\, s}}}, \quad s < -\log\cosh\delta / \sinh\delta$$

$$\lambda(s) =$$

$$-\frac{1}{\sinh\delta}\operatorname{arctanh}\left(\sqrt{1-\cosh^2\delta\, e^{2\sinh\delta\, s}}\right) + \arctan\left(\frac{\sqrt{1-\cosh^2\delta\, e^{2\sinh\delta\, s}}}{\sinh\delta}\right)$$



New spherical curves III:

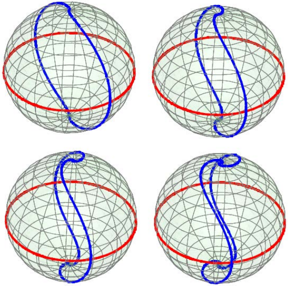
$$\boxed{\kappa(z) = \frac{p(1-2z^2)}{\sqrt{1-z^2}} = \frac{p\cos 2\varphi}{\cos\varphi} = \kappa(\varphi)}, \quad 0 < p < 1$$

$$\mathcal{K}(z) = p\,z\sqrt{1-z^2} = \tfrac{p}{2}\sin 2\varphi = \mathcal{K}(\varphi)$$

$$\varphi(s) = \operatorname{am}(s, p)$$

$$\kappa(s) = p(2\operatorname{cn}(s,p) - 1/\operatorname{cn}(s,p))$$

$$\lambda(s) = -\tfrac{p}{2p'}\log\left(\tfrac{\operatorname{dn}(s,p)+p'}{\operatorname{dn}(s,p)-p'}\right), \quad p' = \sqrt{1-p^2}$$



Uniqueness results on classical spherical curves

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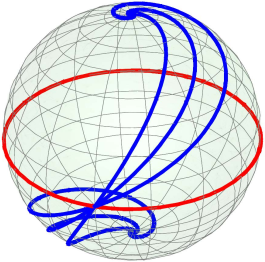
Loxodromes



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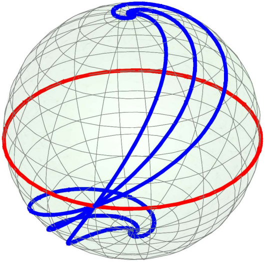

The loxodromes, $d\lambda = \cot\alpha \frac{d\varphi}{\cos\varphi}$, $\alpha \in (0, \pi/2)$, are the only spherical curves (up to rotations around z-axis) with spherical angular momentum $\mathcal{K}(\varphi) = -\cos\alpha \cos\varphi$ (and curvature $\boxed{\kappa(\varphi) = \cos\alpha \tan\varphi}$).

$$\kappa(s) = \cos\alpha \, \tan(\sin\alpha \, s), \; \alpha \in (0, \pi/2)$$

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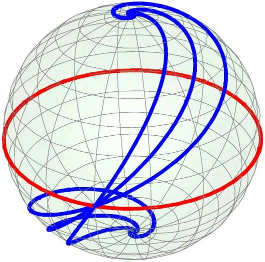

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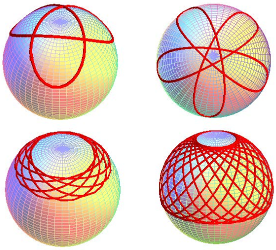
Spherical catenaries



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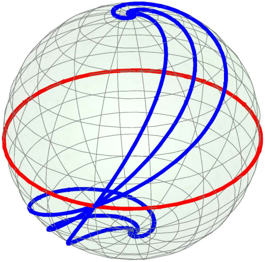

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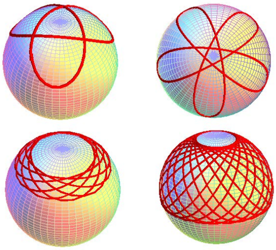

The spherical catenaries, $\sin\varphi\cos^2\varphi\,\frac{d\theta}{ds} = a$, $a < 1/2$, are the only spherical curves (up to rotations around z-axis) with spherical angular momentum
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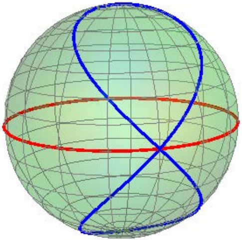
$$\kappa(s) = \frac{2a}{1+\sqrt{1-4a^2}\sin 2s}$$

Uniqueness results on classical spherical curves

Viviani's curve



Uniqueness results on classical spherical curves

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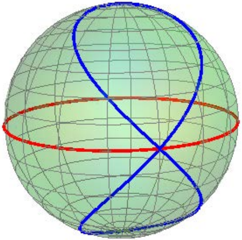

Viviani's curve, $\lambda = \varphi$, is the only spherical curve (up to rotations around z-axis) with spherical angular momentum $\mathcal{K}(z) = \frac{z^2-1}{\sqrt{2-z^2}}$

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Uniqueness results on classical spherical curves

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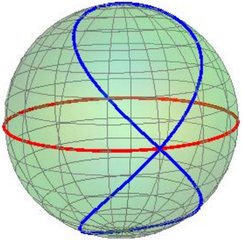

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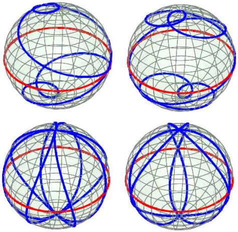
Archimedean spherical spirals



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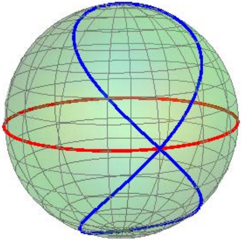

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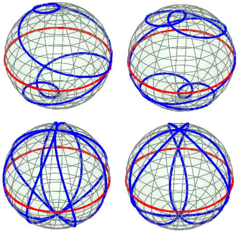

Archimedean spherical spirals, $\varphi = n\lambda$, $n > 0$, are the only spherical curves (up to rotations around z-axis) with spherical angular momentum $\mathcal{K}(z) = \frac{z^2-1}{\sqrt{1+n^2-z^2}}$

(and curvature $\boxed{\kappa(z) = \frac{z(2n^2+1-z^2)}{(n^2+1-z^2)^{3/2}}}$).

Index

Curves in Lorentz-Minkowski plane

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- $\gamma = (x, y)$ *unit-speed* spacelike (resp. timelike) if $g(\dot{\gamma}(s), \dot{\gamma}(s)) = \epsilon$, $\forall s \in I$ ($\epsilon = 1$ if γ spacelike, $\epsilon = -1$ if γ timelike)
- $T = \dot{\gamma} = (\dot{x}, \dot{y})$, $N = \dot{\gamma}^{\perp} = (\dot{y}, \dot{x})$, $g(T, T) = \epsilon$, $g(N, N) = -\epsilon$

Frenet frame and eqns: $\dot{T}(s) = \kappa(s) N(s)$, $\dot{N}(s) = \kappa(s) T(s)$

Curves in Lorentz-Minkowski plane

$$\mathbb{L}^2 := (\mathbb{R}^2, g = -dx^2 + dy^2)$$

• $0 \neq v \in \mathbb{L}^2$ spacelike if $g(v,v) > 0$, lightlike if $g(v,v) = 0$, and timelike if $g(v,v) < 0$

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Theorem

Prescribe $\kappa = \kappa(s)$:

Any *spacelike* curve $\alpha(s)$ in $\mathbb{L}^2$ can be represented (up to isometries) by

$$\alpha(s) = \left(\int \sinh\varphi(s) ds, \int \cosh\varphi(s) ds \right) \text{ with } \varphi(s) = \int \kappa(s) ds.$$

Any *timelike* curve $\beta(s)$ in $\mathbb{L}^2$ can be represented (up to isometries) by

$$\beta(s) = \left(\int \cosh\phi(s) ds, \int \sinh\phi(s) ds \right) \text{ with } \phi(s) = \int \kappa(s) ds.$$

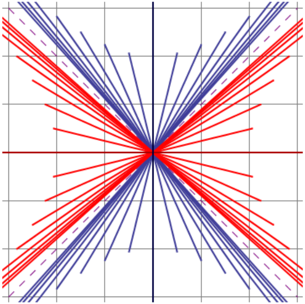
Geodesics

The *spacelike* geodesics can be written as:

$$\alpha_{\varphi_0}(s) = (\sinh \varphi_0 s, \cosh \varphi_0 s), \; s \in \mathbb{R}, \; \varphi_0 \in \mathbb{R},$$

and the *timelike* geodesics can be written as:

$$\beta_{\phi_0}(s) = (\cosh \phi_0 s, \sinh \phi_0 s), \; s \in \mathbb{R}, \; \phi_0 \in \mathbb{R}.$$



Lorentzian Pseudodistance

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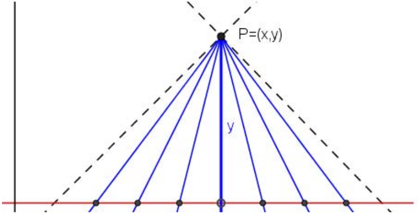

$P = (x, y) \in \mathbb{L}^2$, $y \neq 0$; spacelike geodesics α_m with slope $m = \coth \varphi_0$, $|m| > 1$; $P' = (x - y/m, 0)$

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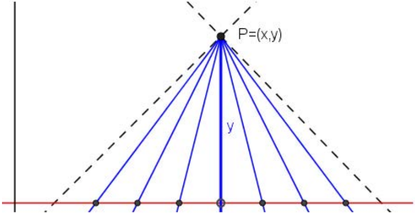

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$$0 < \delta(P, P')^2 = \left(1 - \frac{1}{m^2}\right) y^2 = \frac{y^2}{\cosh^2 \varphi_0} \leq y^2; \text{ “ = ” } \Leftrightarrow \varphi_0 = 0$$

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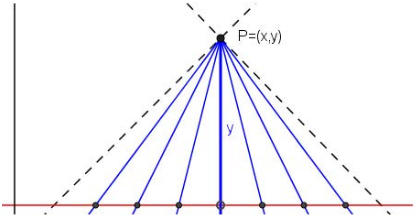

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$|y|$: maximum Lorentzian pseudodistance through spacelike geodesics from $P = (x, y)$, $y \neq 0$, to the x-axis.

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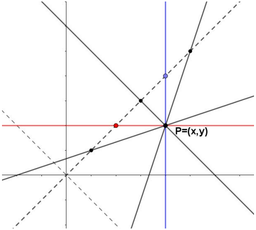

$P = (x, y) \in \mathbb{L}^2$, $x \neq y$; spacelike and timelike geodesics γ_m, $m \in \mathbb{R} \cup \{\infty\}$, $m \neq 1$; $P' = \left(\frac{mx-y}{m-1}, \frac{mx-y}{m-1}\right)$

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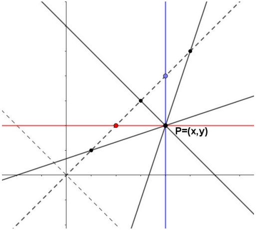

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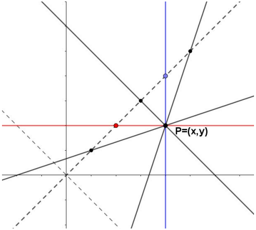

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$|y - x|$: Lorentzian pseudodistance from $P = (x, y) \in \mathbb{L}^2$, $x \neq y$, to the lightlike geodesic $x = y$ through the horizontal timelike geodesic or the vertical spacelike geodesic.

Singer's Problem in $\mathbb{L}^2$

Determine those (spacelike and timelike) curves $\gamma = (x, y)$ in $\mathbb{L}^2$ whose curvature κ depends on some given function $\kappa = \kappa(x, y)$

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$\gamma = (x, y)$ spacelike (resp. timelike) with $\kappa = \kappa(y)$
$\Rightarrow \hat{\gamma} = (y, x)$ timelike (resp. spacelike) with $\kappa = \kappa(x)$

$\kappa(x, y) = \kappa(y)$

Theorem

Prescribe $\kappa = \kappa(y)$ continuous.
Then the problem of determining a spacelike or timelike curve $(x(s), y(s))$ -s arc length- is solvable by three quadratures ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike):

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▶ Such a curve is uniquely determined by $\mathcal{K}(y)$ up to a x-translation.
• $\mathcal{K}(y)$ will distinguish geometrically the curves inside a same family by their relative position with respect to the x-axis.

Example 1

Geodesics: $\kappa \equiv 0$

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- $\mathcal{K}(y) = c \in \mathbb{R}$. $s = \int \frac{dy}{\sqrt{c^2+\epsilon}} = \frac{y}{\sqrt{c^2+\epsilon}}$, $c^2+\epsilon > 0$.

$$x(s) = c\,s,\ y(s) = \sqrt{c^2+\epsilon}\,s,\ s \in \mathbb{R}.$$

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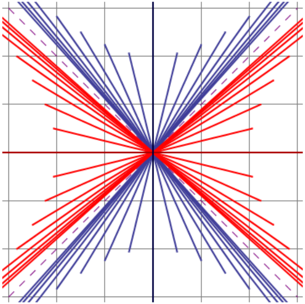
$\epsilon = 1$: $K \equiv c := \sinh \varphi_0 \rightarrow$ spacelike geodesics α_{φ_0}.
$c = 0 = \varphi_0$ corresponds to the y-axis.

$\epsilon = -1$: $K \equiv c := \cosh \phi_0 \rightarrow$ timelike geodesics β_{ϕ_0}.
$c = 1 \Leftrightarrow \phi_0 = 0$ corresponds to the x-axis.



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$\epsilon = 1$: $s = \mathrm{arcsinh}(k_0 y + c)/k_0$.

$x(s) = \cosh(k_0 s)/k_0$, $y(s) = (\sinh(k_0 s) - c)/k_0$, $s \in \mathbb{R}$.

$\epsilon = -1$: $s = \mathrm{arccosh}(k_0 y + c)/k_0$

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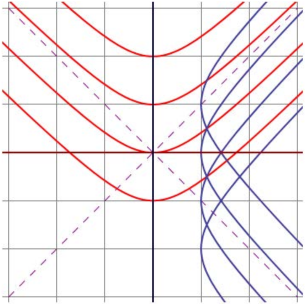

Spacelike and timelike pseudocircles in $\mathbb{L}^2$ of radius $1/k_0$.

Elasticae in $\mathbb{L}^2$

Definition

γ, spacelike or timelike curve in $\mathbb{L}^2$, *elastica under tension* σ if $2\ddot{\kappa} - \kappa^3 - \sigma\kappa = 0$, $\sigma \in \mathbb{R}$. *Energy* $E := \dot{\kappa}^2 - \frac{1}{4}\kappa^4 - \frac{\sigma}{2}\kappa^2$.

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$$\kappa(y) = 2ay + b,\ a \neq 0,\ b \in \mathbb{R}$$

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Proposition

γ spacelike or timelike curve in $\mathbb{L}^2$

(i) If $\kappa(y) = 2ay + b$, $a \neq 0$, $b \in \mathbb{R}$, and $\mathcal{K}(y) = ay^2 + by + c$, $a \neq 0$, $b, c \in \mathbb{R}$, then γ elastica under tension $\sigma = 4ac - b^2$ and energy $E = 4\epsilon a^2 + \sigma^2/4$ (where $\epsilon = 1$ if γ is spacelike and $\epsilon = -1$ if γ is timelike).

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(ii) If γ elastica under tension σ and energy E, with $E \neq \sigma^2/4$, then $\kappa(y) = 2ay + b$, $a \neq 0$, $b \in \mathbb{R}$.

Spacelike elasticae: $\boxed{\kappa(y) = 2y,\ \epsilon = 1}$

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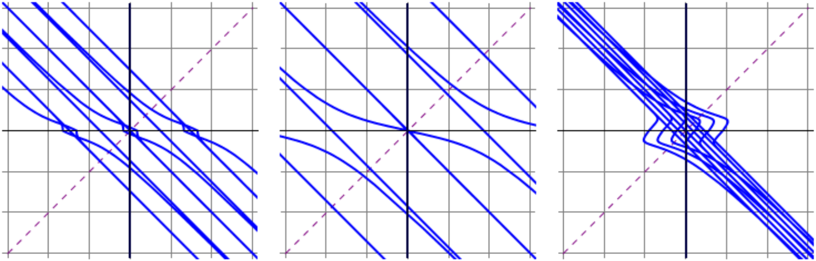
$$x_\eta(s) = (s_\eta + c_\eta)s + \sqrt{c_\eta}\left(\mathrm{cn}(\sqrt{c_\eta}\,s, k_\eta)\left(k_\eta^2\,\mathrm{sd}(\sqrt{c_\eta}\,s, k_\eta) - \mathrm{ds}(\sqrt{c_\eta}\,s, k_\eta)\right) - 2E(\sqrt{c_\eta}\,s, k_\eta)\right)$$

$$y_\eta(s) = \sqrt{c_\eta}\ \mathrm{cs}(\sqrt{c_\eta}\,s, k_\eta)\,\mathrm{nd}(\sqrt{c_\eta}\,s, k_\eta),\ \ k_\eta^2 = \frac{1-\tanh\eta}{2}$$

$$s \in (2mK(k_\eta)/\sqrt{c}_\eta, 2(m+1)K(k_\eta)/\sqrt{c}_\eta),\ m \in \mathbb{N}$$

$$\kappa_\eta(s) = 2\sqrt{c_\eta}\ \mathrm{cs}(\sqrt{c_\eta}\,s, k_\eta)\,\mathrm{nd}(\sqrt{c_\eta}\,s, k_\eta).$$



Spacelike elastic curves $\alpha_\eta = (x_\eta, y_\eta)$, $(\eta = 0, 1{,}5, -1{,}5)$.

Timelike elasticae: $\boxed{\kappa(y) = 2y,\ \epsilon = -1}$

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- $\mathcal{K}(y) = y^2 + 1$ $(c = 1)$.

$x_1(s) = s - \sqrt{2}\coth(\sqrt{2}s)$,

$y_1(s) = -\frac{\sqrt{2}}{\sinh(\sqrt{2}s)}$, $s \neq 0$.

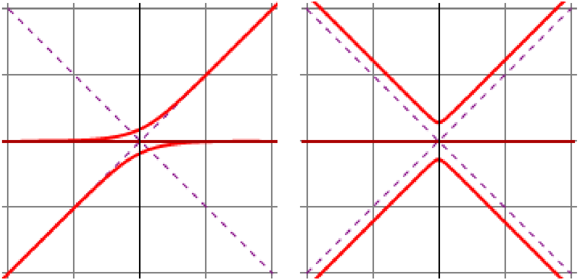
$\kappa_1(s) = -\frac{2\sqrt{2}}{\sinh(\sqrt{2}s)}$.

- $\mathcal{K}(y) = y^2 - 1$ $(c = -1)$.

$x_{-1}(s) = \sqrt{2}\tan(\sqrt{2}s) - s$,

$y_{-1}(s) = \pm\frac{\sqrt{2}}{\cos(\sqrt{2}s)}$, $|s| < \frac{\pi}{2\sqrt{2}}$.

$\kappa_{-1}(s) = \frac{\mp 2\sqrt{2}}{\cos(\sqrt{2}s)}$.



Timelike elasticae: $\boxed{\kappa(y) = 2y,\ \epsilon = -1}$

- $\mathcal{K}(y) = y^2 + \cosh^2 \delta$, $\delta > 0$ ($c > 1$)

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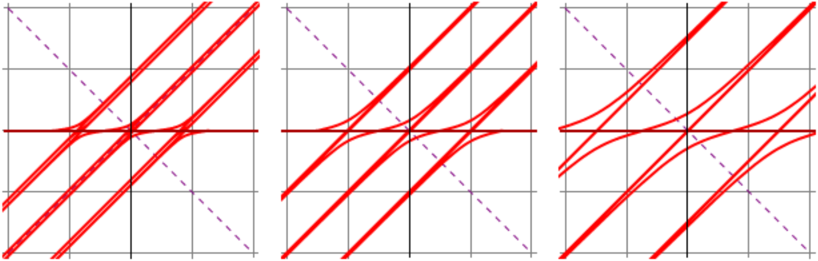
- $\mathcal{K}(y) = y^2 + \cosh^2\delta,\ \delta > 0$ $(c > 1)$

$x_\delta(s) = c_\delta^2 s + \sqrt{c_\delta^2+1}\left(\mathrm{dn}(\sqrt{c_\delta^2+1}\,s, k_\delta)\,\mathrm{tn}(\sqrt{c_\delta^2+1}\,s, k_\delta) - E(\sqrt{c_\delta^2+1}\,s, k_\delta)\right),$

$y_\delta(s) = s_\delta\ \mathrm{tn}(\sqrt{c_\delta^2+1}\,s, k_\delta),\ \ k_\delta^2 = \frac{2}{1+\cosh^2\delta},$

$s \in \left((2m-1)K(k_\delta)/\sqrt{c_\delta^2+1}, (2m+1)K(k_\delta)/\sqrt{c_\delta^2+1}\right),\ m \in \mathbb{N}.$

$\kappa_\delta(s) = 2s_\delta\ \mathrm{tn}(\sqrt{c_\delta^2+1}\,s, k_\delta).$



Timelike elastic curves $\beta_\delta = (x_\delta, y_\delta)$ $(\delta = 0{,}5, 1, 1{,}5)$.

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$x_\psi(s) = s + \sqrt{2}\left(\mathrm{dn}(\sqrt{2}\,s, k_\psi)\,\mathrm{tn}(\sqrt{2}\,s, k_\psi) - E(\sqrt{2}\,s, k_\psi)\right),$

$y_\psi(s) = \sqrt{1 - s_\psi}\,\mathrm{nc}(\sqrt{2}\,s, k_\psi),\ \ k_\psi^2 = \frac{1+\sin\psi}{2},$

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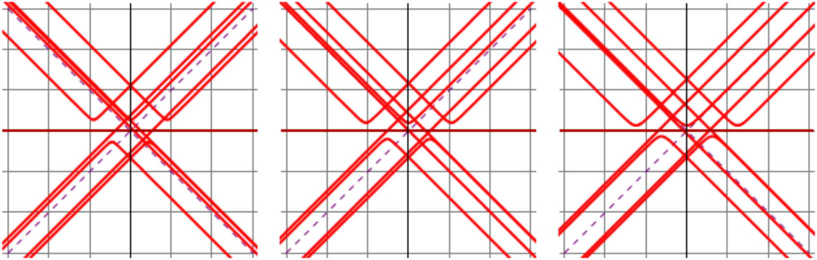

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$$x_\tau(s) = = s + \sqrt{1+c_\tau^2}\left(\mathrm{dn}(\sqrt{1+c_\tau^2}\, s, k_\tau)\,\mathrm{tn}(\sqrt{1+c_\tau^2}\, s, k_\tau) - E(\sqrt{1+c_\tau^2}\, s, k_\tau)\right),$$

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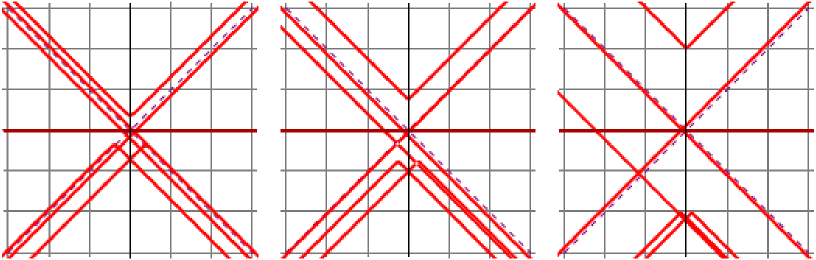

Timelike elastic curves $\beta_\tau = (x_\tau, y_\tau)$, $(\tau = 1, 2, 3)$.

$\kappa(y) = 1/y^2$

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- $\mathcal{K}(y) = -1/y$

$\epsilon = 1$. Spacelike case:

$x(s) = \mp \operatorname{arccosh} s$, $s > 1$.

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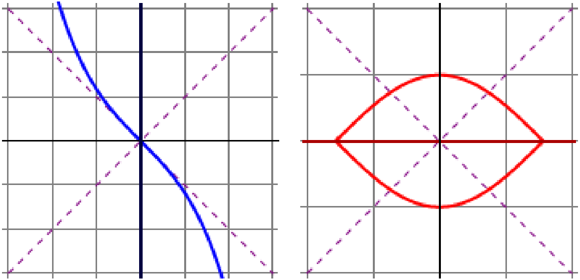

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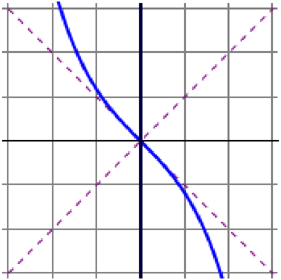
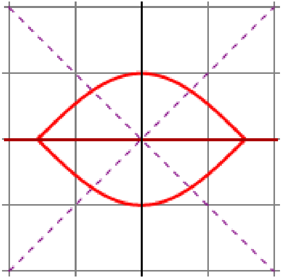
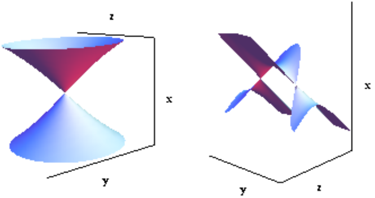




“Lorentzian catenaries”

Lorentzian catenaries.

Kobayashi introduced in 1993, studying maximal rotation surfaces in $\mathbb{L}^3$, the catenoid of the first kind with equation $y^2 + z^2 - \sinh^2 x = 0$ and the catenoid of the second kind with equation $x^2 - z^2 = \cos^2 y$.



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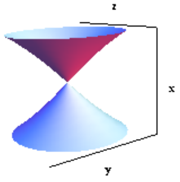

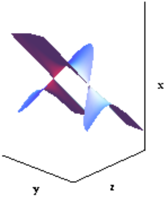

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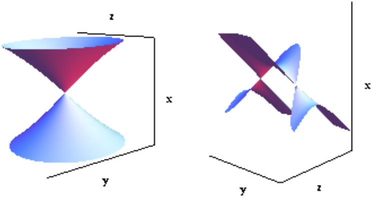

The generatrix curves of both catenoids coincide with the graph $y = -\sinh x$, $x \in \mathbb{R}$ and the bigraph $x = \pm\cos y$, $|y| < \pi/2$.

1. The Lorentzian catenary of the first kind $y = -\sinh x$, $x \in \mathbb{R}$, is the only *spacelike* curve (up to translations in the x-direction) with geometric linear momentum $\mathcal{K}(y) = -1/y$.
2. The Lorentzian catenary of the second kind $x = \pm\cos y$, $|y| < \pi/2$, is the only *spacelike* curve (up to translations in the y-direction) with geometric linear momentum $\mathcal{K}(x) = -1/x$.

$\kappa(y) = 1/y^2$

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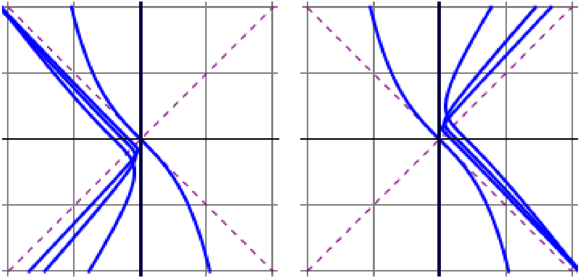
- $\mathcal{K}(y) = c - 1/y$, $c \neq 0$. $\epsilon = 1$, Spacelike case:

$$x = \frac{1}{c^2+1}\left(c\sqrt{(c^2+1)y^2 - 2cy + 1} - \frac{1}{\sqrt{c^2+1}}\operatorname{arcsinh}((c^2+1)y - c)\right).$$



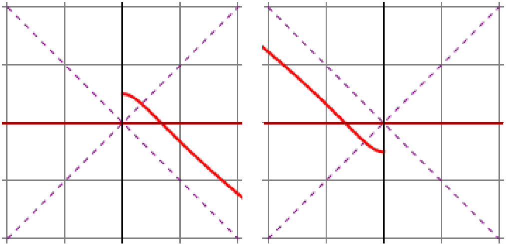
Curves with $\mathcal{K}(y) = c - 1/y$; $c \leq 0$ (left) and $c \geq 0$ (right).

$\kappa(y) = 1/y^2$

- $\mathcal{K}(y) = c - 1/y$, $c \neq 0$. $\epsilon = -1$, Timelike case:

· $\mathcal{K}(y) = 1 - 1/y$:
$x = \frac{(2-y)\sqrt{1-2y}}{3}$, $y < 1/2$.

· $\mathcal{K}(y) = -1 - 1/y$:
$x = -\frac{(2+y)\sqrt{1+2y}}{3}$, $y > -1/2$.



· $\mathcal{K}(y) = c - 1/y$, $|c| > 1$:

$$x = \frac{1}{c^2-1}\left(c\sqrt{(c^2-1)y^2-2cy+1} + \frac{\log\left(2(\sqrt{c^2-1}\sqrt{(c^2-1)y^2-2cy+1}+(c^2-1)y-c)\right)}{\sqrt{c^2-1}}\right).$$

· $\mathcal{K}(y) = c - 1/y$, $|c| < 1$:

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$y(s) = \log(-\operatorname{csch} s)$, $s < 0$.
$\kappa(s) = -\operatorname{csch} s$, $s < 0$.

$y = \log(\sinh x)$, $x > 0$.

$\epsilon = -1$. Timelike case:
$x(s) =$
$\log(\sec s + \tan s)$, $|s| < \pi/2$.
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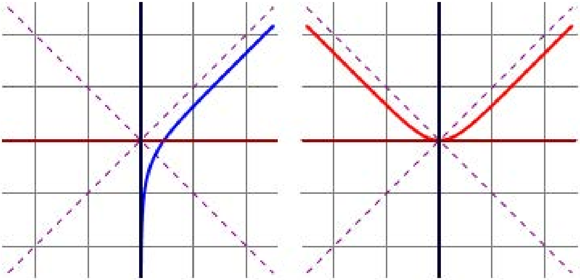

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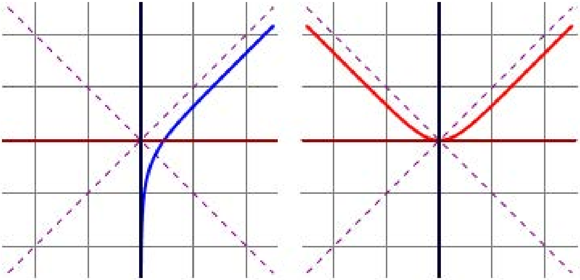
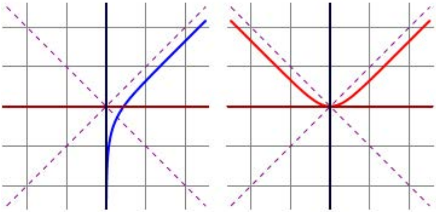


“Lorentzian grim-reapers”

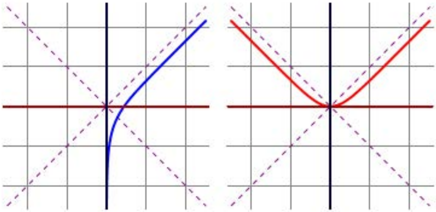
Lorentzian grim-reapers



Both curves satisfy the translating-type soliton equation:

$$\kappa = g((0,1), N)$$

Lorentzian grim-reapers



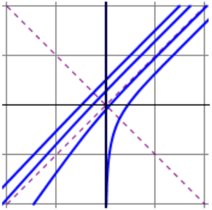
Both curves satisfy the translating-type soliton equation:

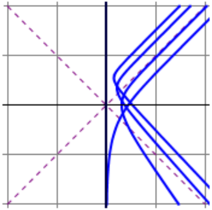
$$\kappa = g((0,1), N)$$

1. The Lorentzian grim-reaper $y = \log(\sinh x)$, $x > 0$, is the only *spacelike* curve (up to x-translations) in $\mathbb{L}^2$ with geometric linear momentum $\mathcal{K}(y) = e^y$.
2. The Lorentzian grim-reaper $y = \log(\cosh x)$, $x \in \mathbb{R}$, is the only *timelike* curve (up to x-translations) in $\mathbb{L}^2$ with geometric linear momentum $\mathcal{K}(y) = e^y$.

$\kappa(y) = e^y$

- $\mathcal{K}(y) = e^y + c,\ c \neq 0.$





Spacelike case ($\epsilon = 1$):

$x = \operatorname{arcsinh}(e^y + c) - \frac{c}{\sqrt{c^2+1}} \operatorname{arcsinh}\left(c + (c^2+1)e^{-y}\right).$

Timelike case ($\epsilon = -1$):

$\cdot \mathcal{K}(y) = e^y + 1$:

$x = 2\log(\sqrt{e^y} + \sqrt{e^y + 2}) - \sqrt{1 + 2e^{-y}}.$

$\cdot \mathcal{K}(y) = e^y + c,\ |c| > 1$:

$x = \log\left(2(\sqrt{P(e^y)} + e^y + c)\right) - \frac{c \log\left(2e^{-y}(\sqrt{c^2-1}\sqrt{P(e^y)} + ce^y + c^2 - 1)\right)}{\sqrt{c^2-1}}$

$\cdot \mathcal{K}(y) = e^y - 1$:

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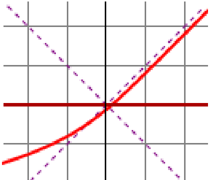
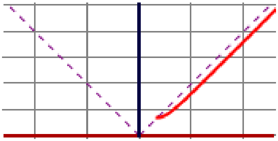
$\cdot \mathcal{K}(y) = e^y + c,\ |c| < 1$:

$x = \log\left(2(\sqrt{P(e^y)} + e^y + c)\right) + \frac{c}{\sqrt{1-c^2}} \arcsin\left(c + (c^2 - 1)e^{-y}\right).$





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Theorem

Prescribe $\kappa = \kappa(v)$ continuous.
Then the problem of determining a spacelike or timelike curve

$$\left(\frac{u(s)-v(s)}{2}, \frac{u(s)+v(s)}{2}\right)$$

-s arc length- is solvable by three quadratures
($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike):

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▶ Such a curve is uniquely determined by $\mathcal{K}(v)$ up to a u-translation.

• $\mathcal{K}(v)$ will distinguish geometrically the curves inside a same family by their relative position with respect to the u-axis.

Examples: constant curvature

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Geodesics: $\kappa \equiv 0$

• $\mathcal{K}(v) = -\epsilon/c$, $c \neq 0$. $u(s) = -\epsilon s/c$, $v(s) = -cs$, $s \in \mathbb{R}$
(lines passing through the origin with slope $m = \frac{\epsilon + c^2}{\epsilon - c^2}$).
$\epsilon = 1 \Rightarrow |m| > 1$ spacelike geodesics, $\epsilon = -1 \Rightarrow |m| < 1$ timelike geodesics.

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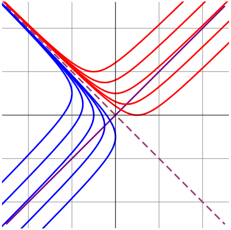
$\epsilon = 1 \Rightarrow |m| > 1$ spacelike geodesics, $\epsilon = -1 \Rightarrow |m| < 1$ timelike geodesics.

Circles: $\kappa \equiv k_0 > 0$

- $\mathcal{K}(v) = \frac{-\epsilon}{c+k_0 v}$, $c \in \mathbb{R}$. $u(s) = -\epsilon e^{k_0 s}/k_0$, $v(s) = (e^{-k_0 s} - c)/k_0$.

$\epsilon = 1 \Rightarrow x(s) = (-\cosh(k_0 s) + c/2)/k_0$, $y(s) = -(\sinh(k_0 s) + c/2)/k_0$.

$\epsilon = -1 \Rightarrow x(s) = (\sinh(k_0 s) + c/2)/k_0$, $y(s) = (\cosh(k_0 s) - c/2)/k_0$.



Spacelike and timelike pseudocircles in $\mathbb{L}^2$ of radius $1/k_0$.

$\kappa(v) = 2v$

σ-elastica: $2\ddot{\kappa} - \kappa^3 - \sigma\kappa = 0$, $\sigma \in \mathbb{R}$.

Energy $E \in \mathbb{R}$: $E = \dot{\kappa}^2 - \frac{1}{4}\kappa^4 - \frac{\sigma}{2}\kappa^2$.

$\boxed{E = \sigma^2/4 \ ?}$

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- $\mathcal{K}(v) = -\frac{\epsilon}{v^2+c}$, $c \in \mathbb{R}$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike).

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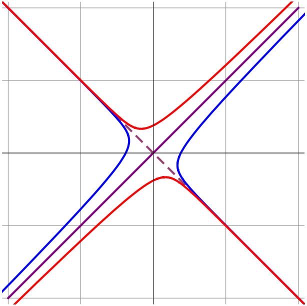
- $\mathcal{K}(v) = -\frac{\epsilon}{v^2+c},\ c \in \mathbb{R}$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike).

1. $c = 0$: $u(s) = -\epsilon s^3/3,\ v(s) = 1/s,\quad \kappa(s) = 2/s,\ s \neq 0$.



Spacelike (blue) and timelike (red) elastic curve with $\sigma = E = 0$

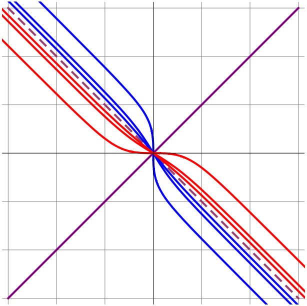
$\kappa(v) = 2v$

σ-elastica: $2\ddot{\kappa} - \kappa^3 - \sigma\kappa = 0$, $\sigma \in \mathbb{R}$.

Energy $E \in \mathbb{R}$: $E = \dot{\kappa}^2 - \frac{1}{4}\kappa^4 - \frac{\sigma}{2}\kappa^2$. $\boxed{E = \sigma^2/4\ ?}$

- $\mathcal{K}(v) = -\frac{\epsilon}{v^2+c}$, $c \in \mathbb{R}$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike).

2. $c > 0$: $u(s) = -\frac{\epsilon}{c}\left(\frac{s}{2} + \frac{\sin(2\sqrt{c}s)}{4\sqrt{c}}\right)$, $v(s) = -\sqrt{c}\tan(\sqrt{c}s)$.
   $\kappa(s) = -2\sqrt{c}\tan(\sqrt{c}s)$, $|s| < \pi/2\sqrt{c}$.



Spacelike (blue) and timelike (red) elastic curves
with $\sigma = 4c > 0$ and $E = 4c^2$ ($c = 1, 2, 3$)

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σ-elastica: $2\ddot{\kappa} - \kappa^3 - \sigma\kappa = 0$, $\sigma \in \mathbb{R}$.

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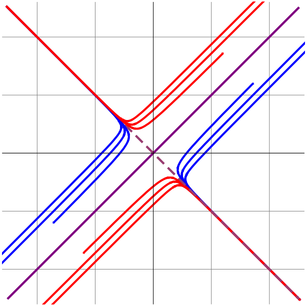
3. $c < 0$: $u(s) = \frac{\epsilon}{c}\left(-\frac{s}{2} + \dfrac{\sinh(2\sqrt{-c}\,s)}{4\sqrt{-c}}\right)$, $v(s) = \sqrt{-c}\coth(\sqrt{-c}\,s)$.
   $\kappa(s) = 2\sqrt{-c}\coth(\sqrt{-c}s)$, $s \neq 0$.



Spacelike (blue) and timelike (red) elastic curves with $\sigma = 4c < 0$ and $E = 4c^2$ ($c = -1, -2, -3$)

$\kappa(v) = 1/v^2$

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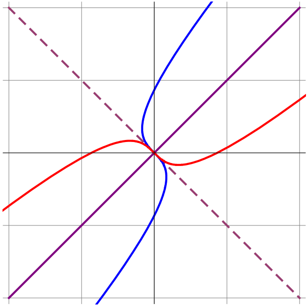
- $\mathcal{K}(v) = \epsilon v$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

$u(s) = 2\epsilon\sqrt{2s}\sqrt{s}/3, \quad v(s) = \sqrt{2s}, \qquad \kappa(s) = \dfrac{1}{2s}, \; s > 0.$

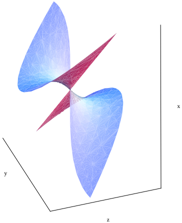
Graphs $u = \epsilon\, v^3/3$, $v > 0$ for $\epsilon = \pm 1$.



Spacelike (blue) and timelike (red) curve in $\mathbb{L}^2$ with $\mathcal{K}(v) = \epsilon v$, $\epsilon = \pm 1$.

Generatrix of Enneper's surface of second kind

Kobayashi, 1993: Enneper's surface of second kind.
Rotation surface with lightlike axis $(1, 0, 1)$ and generatrix curve $x = \lambda(-t + t^3/3)$, $z = \lambda(t + t^3/3)$, $\lambda > 0$, at the xz-plane.



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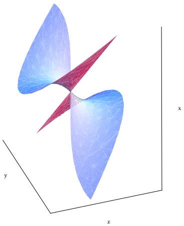

The generatrix curve of Enneper's surface for $\lambda = 1/2$ coincide with the graph $u = v^3/3$, $v > 0$.

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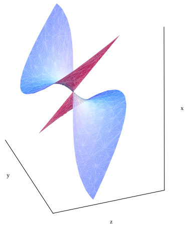

The generatrix curve of Enneper's surface for $\lambda = 1/2$ coincide with the graph $u = v^3/3$, $v > 0$.

The generatrix curve of the Enneper's surface of second kind, $u = v^3/3$, $v > 0$, is the only *spacelike* curve (up to dilations and u-translations) with geometric linear momentum $\mathcal{K}(v) = v$ (and curvature $\kappa(v) = 1/v^2$)

$\kappa(v) = 1/v^2$

- $\mathcal{K}(v) = \frac{-\epsilon v}{c\,v-1}$, $c \neq 0$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

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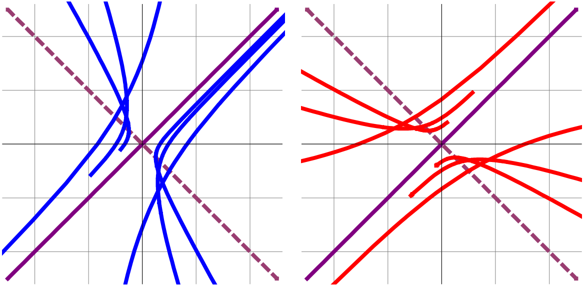
- $\mathcal{K}(v) = \frac{-\epsilon v}{c\,v-1}$, $c \neq 0$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

$$u = u(v) = \frac{\epsilon}{c^3}\left(c\,v - 1 - \frac{1}{c\,v-1} + 2\log(c\,v-1)\right),$$
$$v > 1/c \text{ if } c > 0,\; v < 1/c \text{ if } c < 0.$$



Spacelike curves with $\mathcal{K}(v) = -\frac{v}{c\,v-1}$ (left) and timelike curves with $\mathcal{K}(v) = \frac{v}{c\,v-1}$ (right).

$$\kappa(v) = e^v$$

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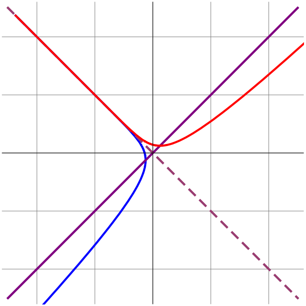
$\kappa(v) = e^v$

- $\mathcal{K}(v) = -\frac{\epsilon}{e^v + c}$, $c \in \mathbb{R}$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)
  1. $c = 0$: $u(s) = -\epsilon s^2/2$, $v(s) = -\log s$, $\kappa(s) = 1/s$, $s > 0$.

Graph $u = -\epsilon\, e^{-2v}/2$, $v \in \mathbb{R}$.

Translating-type soliton equation: $\kappa = g((1,1), N)$.



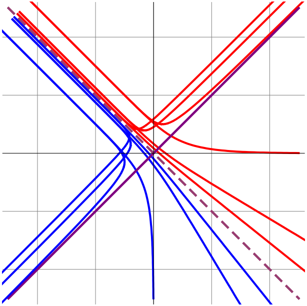
Spacelike (blue) and timelike (red)

Lorentzian grim-reapers, $\mathcal{K}(v) = -\frac{\epsilon}{e^v}$

$\kappa(v) = e^v$

- $\mathcal{K}(v) = -\frac{\epsilon}{e^v + c}$, $c \in \mathbb{R}$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

2. $c \neq 0$: $u(s) = -\frac{\epsilon}{c}\left(s + \frac{1}{c\,e^{cs}}\right)$, $v(s) = \log \frac{c}{e^{cs} - 1}$,
   $\kappa(s) = \frac{c}{e^{cs} - 1}$, $s > 0$



Spacelike curves (blue) and timelike curves (red)
with $\mathcal{K}(v) = -\frac{\epsilon}{e^v + c}$, $c \neq 0$

References

I. Castro, I. Castro-Infantes and J. Castro-Infantes: *Curves in Lorentz-Minkowski plane: elasticae, catenaries and grim-reapers*. Open Math. **16** (2018), 747–776.

I. Castro, I. Castro-Infantes and J. Castro-Infantes: *Curves in Lorentz-Minkowski plane with curvature depending on their position*. Preprint 2018. arXiv:1806.09187 [math.DG].

References

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Thank you very much
for your attention!