On Rotary Mappings and Transformations

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Introduction

A special diffeomorphism between (pseudo-) Riemannian manifolds and manifolds with affine and projective connections, for which maps any special curve onto a special curve, were studied in many works.

For example geodesic mappings, for which any geodesic maps onto geodesic.
Analogically holomorphically-projective and F-planar mappings for which any analytic and F-planar curve maps onto analytic and F-planar curve, respectively and
almost geodesic mapping is define as, for any geodesic maps onto almost geodesic curve, see [9].[1]

[1] [9] Mikeš J. et al., *Differential geometry of special mappings*, Palacky Univ. Press, Olomouc 2015.

Rotary diffeomorphism

Leiko [3][2] was the first one to introduce term of rotary mapping.

Definition

A diffeomorphism between two-dimensional (pseudo-) Riemannian manifolds is called *rotary* if any geodesic is mapped onto isoperimetric extremal of rotation.

[2] [3] Leiko, S. Rotary diffeomorphisms on Euclidean spaces. *Mat. Zametki*, 47(3), 1990, p. 52–57. (Math. Notes)

Isoperimetric extremal of rotation

Leiko [3][3] was the first one to introduce term of isoperimetric extremals of rotation on two-dimensional Riemannian spaces $\mathbb{V}_2$ and surfaces $\mathcal{S}_2$ with metric g.

Definition

A curve ℓ: $x = x(t)$ on surface or on two-dimensional (pseudo-) Riemannian space is called the

isoperimetric extremal of rotation

if ℓ is extremal of functionals $\theta[\ell]$ and $s[\ell] = \text{const}$ with fixed ends. Here

$$s[\ell] = \int_{t_0}^{t_1} |\lambda| \, \mathrm{d}t \quad \text{and} \quad \theta[\ell] = \int_{t_0}^{t_1} k(t) \, \mathrm{d}t,$$

where $k(t)$ is the curvature and

$|\lambda|$ is the length of the tangent vector λ of ℓ.

[3] [3] Leiko, S. Rotary diffeomorphisms on Euclidean spaces. *Mat. Zametki*, 47(3), 1990, p. 52–57. (Math. Notes)

Isoperimetric extremal of rotation

We obtain simply equation of IER [11][4]:

A curve ℓ: $x = x(t)$ is an isoperimetric extremal of rotation if and only if the following equation holds $\nabla_s \lambda = cK \cdot F\lambda$, and in the coordinate form:

$$d\lambda^h/dt + \Gamma^h_{\alpha\beta}(x)\lambda^\alpha\lambda^\beta = cK(x) \cdot F^h_\alpha(x)\lambda^\alpha, \tag{1}$$

where c is a constant, K is the Gaussian curvature, $\lambda(t) = dx(t)/dt$ is a tangent vector of ℓ, and F is an affinor, that satisfies

$$F^h_j = g^{hi}\varepsilon_{ij}, \qquad \varepsilon_{ij} = \sqrt{|g_{11}g_{22} - g^2_{12}|} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In this case, parameter t is a length of the curve ℓ and λ is a unit tangent vector.

[4][11] Mikeš, J., Sochor, M., Stepanova, E. On the existence of isoperimetric extremals of rotation and the fundamental equations of rotary diffeomorphisms. Filomat 29:3, 2015, p. 517–523.

On the existence of isoperimetric extremals of rotation

Theorem

Let $\mathbb{V}_2$ be a (non flat) Riemannian manifold of the smoothness class C^3. Then there is precisely one isoperimetric extremal of rotation going through a point $x_0 \in \mathbb{V}_2$ in a given non-isotropic direction $\lambda_0 \in T\mathbb{V}_2$ and constant c.

Proof: Let us write equation (1) as a system of ordinary differential equations:

$$\dot{x}^h(s) = \lambda^h(s)$$

$$\dot{\lambda}^h(s) = -\Gamma^h_{ij}(x(s)) \cdot \lambda^i(s) \cdot \lambda^j(s) + c \cdot K(x(s)) \cdot F^h_i(x(s)) \cdot \lambda^i(s).$$

From the theory of differential equations it is known that given the initial conditions $x^h(0) = x^h_0$ and $\lambda^h(0) = \dot{x}^h(0) = \lambda^h_0$ the system (1) has only one solution when $\Gamma^h_{ij} \in C^1, K \in C^1 \text{ and } F^h_i \in C^1$. These conditions are met on a space $\mathbb{V}_2 \in C^3$ (we consider that $\mathbb{V}_2$ is a metric of some surface $\mathcal{S}_2 \subset \mathbb{E}_3$ of the smoothness class C^4). □

Rotary diffeomorphism

Generalized definition:

Let $V_2 = (M, g)$ be a two-dimensional (pseudo-) Riemannian manifold M with a metric g and $\bar{A}_2 = (\bar{M}, \bar{\nabla})$ be a two-dimensional manifold $\bar{M}$ with an affine connection $\bar{\nabla}$.

If the definition was formulated:

A diffeomorphism between two-dimensional (pseudo-) Riemannian manifolds $\mathbb{V}_2$ and $\bar{\mathbb{V}}_2$ ($\bar{\mathbb{A}}_2$) is called rotary if any isoperimetric extremal of rotation on $\mathbb{V}_2$ is mapped onto geodesic on $\bar{\mathbb{V}}_2$ ($\bar{\mathbb{A}}_2$),

then this mapping would be a geodesic mapping.

Rotary diffeomorphism

Then generalized definition [1][5]:

Definition

A diffeomorphism $f\colon \mathbb{V}_2 \to \bar{\mathbb{A}}_2$ is called *rotary mapping* if any geodesic on manifold $\bar{\mathbb{A}}_2$ with affine connection $\bar{\nabla}$ is mapped onto isoperimetric extremal of rotation on two-dimensional (pseudo-) Riemanninan manifold $\mathbb{V}_2$.

Later, some new properties were proved, see [1]:
When $\mathbb{V}_2$ admits rotary mapping f onto $\bar{\mathbb{A}}_2$ then if $\mathbb{V}_2$ and $\bar{\mathbb{A}}_2$ in common coordinate system belong differentiability class C^2 and C^1, respectively, then Gaussian curvature K on $\mathbb{V}_2$ is differentiable. As a result they formulated new theorem:

[5] [1] Chudá, H., Mikeš, J., Sochor, M. Rotary diffeomorphism onto manifolds with affine connection. In: *Geometry, Integrability and Quantization 18, Proc.18th Int. Conf.* Sofia: Bulgaria, 2017, p. 130–137.

Certain properties

Theorem

Rotary diffeomorphism $\mathbb{V}_2 \to \bar{\mathbb{A}}_2$ does not exist if Gaussian curvature $K \notin C^1$.

Chudá, Mikeš and Sochor [1] later proved that (pseudo-) Riemannian manifold $\mathbb{V}_2$ admits rotary mapping onto $\bar{\mathbb{A}}_2$ if and only if in $\mathbb{V}_2$ holds equation

$$\theta^h{}_{,j} = \theta^h(\theta_j + \partial_j \ln|K|) + \nu\,\delta^h_j, \tag{2}$$

where $\theta_i = g_{i\alpha}\theta^\alpha$, ν is a function on $\mathbb{V}_2$ and vector field θ^h is a special case of torse-forming field.
Here and after comma denotes covariant derivative respective connection ∇, and $\partial_i = \partial/\partial x^i$.

Rotary Mappings of Surfaces of Revolution

Leiko [3][6] has studied rotary mappings of surfaces of revolution. Here, he used the metric of the surface of revolution $\mathcal{S}_2$ in the following form

$$\mathrm{d}s^2 = f(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2. \tag{3}$$

Leiko analyzed the equations $\theta^h{}_{,j} = \theta^h(\theta_j + \partial_j \ln|K|) + \nu\,\delta^h_j$ and proved a theorem that vector fields $\theta^h{}_{,j}$ exist in Riemannian space $\mathbb{V}_2$ if and only if $\mathbb{V}_2$ is isometric with surface of revolution $\mathcal{S}_2$ and the metrics of $\mathbb{V}_2$ has one from the following forms:

$$(\tilde{g}_{ij}) = \frac{f(r)}{A^2(B+\sqrt{f(r)})^2}\ \mathrm{diag}(f(r), r^2),$$

$$(\tilde{g}_{ij}) = B^2 f(r)\,\mathrm{diag}(f(r), r^2), \quad \text{where } A \neq 0 \text{ and } B \text{ are } \mathrm{const}.$$

Above mentioned was formulated in Theorem 2 in [3].

[6] [3] Leiko, S. Rotary diffeomorphisms on Euclidean spaces. Math. Notes, 1990.

Let us remind that the metric of the Riemannian space $\mathbb{V}_2$ (that is induced by the surface of revolution $\mathcal{S}_2$) in certain coordinate system can be written in the form

$$ds^2 = \left(dx^1\right)^2 + f(x^1)\left(dx^2\right)^2, \tag{4}$$

where $f\,(\neq 0)$ is a certain function of x^1.

Note that the metric (4) of the surface of revolution $\mathcal{S}_2$ is more general than the metric in form (3), which was used by Leiko. The metric (4) also includes gorge circles, which are in (3) basically excluded.

Note that existence of coordinate system (4) is connected with existence of anisotropic concircular vector field λ which is characterized by the equations

$$\lambda^h_{,i} = \rho\,\delta^h_i,$$

where ρ is a certain function on $\mathbb{V}_n$.

Locally, $\mathbb{V}_2$ with metric (4) realises as surface of revolution in Euclidean space $\mathbb{E}_3$ given by the equations

$$x = F(x^1)\cos x^2, \quad y = F(x^1)\sin x^2, \quad z = z(x^1), \quad \text{here } f = F^2.$$

In case the metric $\mathrm{d}s^2$ is indefinite the surface is given by the equations

$$x = F(x^1)\cosh x^2, \quad y = F(x^1)\sinh x^2, \quad z = z(x^1),$$

where (x, y, z) are coordinates in Minkowski space, which metric has the form $\mathrm{d}s^2 = \mathrm{d}x^2 - \mathrm{d}y^2 + \mathrm{d}z^2$ therefore for our example

$$\mathrm{d}s^2 = (F'^2 + z'^2)\left(\mathrm{d}x^1\right)^2 - F^2\left(\mathrm{d}x^2\right)^2.$$

Further, we are going to prove that any surface of revolution admits rotary mapping. Moreover, *any Riemannian space* $\mathbb{V}_2$ *that is isometric with such surface of revolution* $\mathcal{S}_2$*, and also any pseudo-Riemannian space* $\mathbb{V}_2$ *with metric mentioned above admits rotary mapping onto space* $\bar{\mathbb{A}}_2$*.*

Theorem

Any surface of revolution $\mathcal{S}_2$ with differentiable Gaussian curvature K admits rotary mapping onto $\bar{\mathbb{A}}_2$.

In the proof of Theorem 6 the metric is used in the form (4) therefore Theorem holds for any Riemannian space $\mathbb{V}_2$ which is isometric with surface of revolution $\mathcal{S}_2$. Moreover, this Theorem holds even for pseudo-Riemannian spaces which have indefinite metric for which $f(x^1) < 0$. In this case instead of $\sqrt{f(x^1)}$ we write $\sqrt{|f(x^1)|}$.

General solution of (2) in system (4) is:

$$\theta^h = a \cdot \delta_1^h \quad \text{where} \quad a(x^1) = \frac{2K \cdot f}{f' + 2C\sqrt{f}}, \tag{5}$$

this solution depends on one parameter C.

Rotary mappings of Riemannian spaces $\mathbb{V}_2$

In papers [3,4,6,8][7] Leiko claims that from the equations

$$\theta^h_{,j} = \theta^h(\theta_j + \partial_j \ln |K|) + \nu\, \delta^h_j \qquad (6)$$

it yields spaces $\mathbb{V}_2$ are isometric with surfaces of revolution.
We proved that above mentioned statement is not valid, i.e. the following theorem holds[8]

Theorem

There exist (pseudo-) Riemannian spaces $\mathbb{V}_2$ where exists vector field (6) *and which are not isometric with surfaces of revolution.*

[7] Leiko, S.: [3] Rotary diffeomorphisms on Euclidean spaces. *Mat. Notes*, 1990. [4] Variational problems for rotation functionals, and spin-mappings of pseudo-Riemannian spaces. *Sov. Math.*, 1990. [6] Isoperimetric extremals of a turn on surfaces in Euclidean space $\mathbb{E}^3$. *Russ. Math.*, 1996. [8] Isoperimetric problems for rotation functionals of the first and second orders in (pseudo) Riemannian manifolds. *Russ. Math.*, 2005.

[8] [10] Mikeš, J., Rýparová, L., Chudá, H. On theory of rotary mappings. *Math. Notes*, 2018.

Rotary mappings of Riemannian spaces $\mathbb{V}_2$

Riemannian spaces where torse-forming vector fields $(\theta^h{}_{,j} = \theta^h a_j + \nu\,\delta^h_j)$ exist are characterized with metric it the following form

$$ds^2 = (dx^1)^2 + f(x^1, \dots, x^n)\, d\tilde{s}^2,$$

where $d\tilde{s}^2$ is a metric of the $(n-1)$ dimensional (pseudo-) Riemannian space $\tilde{\mathbb{V}}_{n-1}$ and f is a function of all variables.

In our case we suppose that the metric of the two-dimensional (pseudo-) Riemannian space $\mathbb{V}_2$ has the following form

$$\mathrm{d}s^2 = \left(\mathrm{d}x^1\right)^2 + f(x^1, x^2)\cdot\left(\mathrm{d}x^2\right). \tag{7}$$

This coordinate system is called a *semi-geodesic*.

In case the function f is a function of variable x^1 then the space $\mathbb{V}_2$ is isometric with surface of revolution.

We suppose that the component θ^2 in this coordinate system is vanishing, and we can rewrite fundamental equation (2) in the following form

$$\theta^h_{,i} \equiv \partial_i \theta^h + \theta^\alpha \Gamma^h_{\alpha i} = \theta^h(\theta_i + \partial_i \ln|K|) + \nu\, \delta^h_i \tag{8}$$

where K is the Gaussian curvature of the space $\mathbb{V}_2$ and $\theta_i = g_{i\alpha}\theta^i$. From it follows $\theta_1 = \theta^1$, and additionally in chosen coordinate system holds $\theta_2 = 0$.
For indices $(hi) = (12)$ from (8) and after lowering indices we obtain $\partial_2\theta_1 = \theta_1 \cdot \partial_2 K/K$ and after integration we get

$$\theta_1 = \varkappa(x^1)K \tag{9}$$

where $\varkappa$ is a function of variable x^1. Evidently, for $(hi) = (21)$ formula (8) is identity and for $(hi) = (11)$ and (22) we get following equations

$$\nu = \theta_{11} - \theta_1^2 - \theta_1\, \partial_i K/K \quad \text{and} \quad \nu = \tfrac{1}{2}\,\theta_1 \cdot f_1/f.$$

We merge these formulas and obtain following equation

$$\varkappa'/\varkappa - \varkappa \cdot K = \tfrac{1}{2} \cdot f_1/f. \tag{10}$$

We can also calculate Gaussian curvature of the space $\mathbb{V}_2$ and get $K = -1/2\,\partial_1 F - 1/4\,F^2$, where $F = f_1/f$. This can be substituted to the above mentioned equations

$$\partial_1 F = -\frac{1}{2}\,F^2 + \frac{1}{\varkappa}\,F - 2 \cdot \frac{\varkappa'}{\varkappa^2} \tag{11}$$

The equation (11) with a priori given function $\varkappa(x^1)$ is differential equation (type Riccati) for unknown function F, here variable x^2 is a parameter. General solution of this system depends on function of variable x^2, which has the role of the "constant of integration".
Since $\partial_1 \ln|f| = F$, after integration we obtain

$$f = c(x^2) \cdot \exp\left(\int F\,\mathrm{d}x^1\right),$$

where $c(x^2)$ is a differentiable function.

Rotary mappings of Riemannian spaces $\mathbb{V}_2$

Obviously, the function $f = c(x^2) \cdot \exp(\int F \,\mathrm{d}x^1)$ depends, in general case, on both variables x^1 and x^2. From this follows that the space $\mathbb{V}_2$ is not isometric with space of revolution. The space $\mathbb{V}_2$ would be isometric with a surface of revolution if for example the function $c(x^2) = \mathrm{const.}$ We obtain [10][9]

Theorem

Any (pseudo-) Riemannian space with metric in form $ds^2 = \left(dx^1\right)^2 + f(x^1, x^2) \cdot \left(dx^2\right)$, where f is a function that satisfies the following condititons

$$\partial_1 F = -F^2/2 + F/\varkappa - 2 \cdot \varkappa'/\varkappa^2, \quad \text{and} \quad f = c(x^2) \cdot \exp(\textstyle\int F \, dx^1),$$

admits rotary mappings onto spaces with affine connection $\bar{\mathbb{A}}_2$.

[9] [10] Mikeš, J., Rýparová, L., Chudá, H. On theory of rotary mappings. *Math. Notes*, 2018.

Rotary vector field in $\mathbb{V}_2$

Theorem

A two-dimensional (pseudo-) Riemannian manifold $\mathbb{V}_2$ admits rotary vector field θ if and only if the following closed Cauchy type system of PDE's in covariant derivatives has a solution with respect to functions $\theta_i(x)$ and $\nu(x)$:

$$\begin{aligned} \theta_{i,j} = {} & \theta_i(\theta_j + \partial_j K/K) + \nu g_{ij}, \\ \nu_{,i} = {} & \nu(\theta_i - \partial_i K/K) - K\theta_i - \theta_\alpha\theta_\beta g^{\alpha\beta}\partial_i K/K + \theta_i g^{\alpha\beta}\theta_\alpha\partial_\beta K/K. \end{aligned} \tag{12}$$

For initial data (= initial Cauchy conditions) $\theta_i(x_0){=}\theta_i^0$ and $\nu(x_0){=}\nu_0$ where $x_0 \in \mathbb{V}_2 \in C^3$ the system (12) has at most one solution

$$\theta_i(x) \quad \text{and} \quad \nu(x).$$

Further, let us note it follows from the initial conditions that the general solution (12) depends on no more than 3 real parameters. These are for example numbers θ_i^0 and ν_0.

Infinitesimal rotary transformations

An *infinitesimal transformation* of a (pseudo-) Riemannian space V_n is given with respect to the coordinates in this manner

$$\bar{x}^h = x^h + \varepsilon\, \xi^h(x), \tag{13}$$

where x^h are the coordinates of a certain point in $\mathbb{V}_n$ and $\bar{x}^h$ are the coordinates of its image under the infinitesimal transformation, ε is an infinitesimal parameter not depending on x^h, and ξ^h is a displacement vector.

Infinitesimal rotary transformations

If a certain object $\mathcal{A}$ of the space $\mathbb{V}_n$ depends on $x \in \mathbb{V}_n$ but also on the infinitesimal parameter ε, then the *principal part* of the object $\mathcal{A}$ is $\overset{0}{\mathcal{A}}(x) + \overset{1}{\mathcal{A}}(x)\varepsilon$ in the expansion of series with respect to the infinitesimal parameter ε

$$\mathcal{A}(x, \varepsilon) = \overset{0}{\mathcal{A}}(x) + \overset{1}{\mathcal{A}}(x)\varepsilon + \overset{2}{\mathcal{A}}(x)\varepsilon^2 + \dots .$$

For our purposes the curves obtained by the infinitesimal transformation of geodesics satisfy the equations of isoperimetric extremals of rotation (1) under the condition, that we dropped the terms containing higher powers of the infinitesimal parameter ε, i.e. $\varepsilon^2, \varepsilon^3, \dots$.

Infinitesimal rotary transformations

An *infinitesimal transformation* of a (pseudo-) Riemannian space V_n is given with respect to the coordinates in this manner

$$\bar{x}^h = x^h + \varepsilon\,\xi^h(x), \tag{14}$$

where x^h are the coordinates of a certain point in $\mathbb{V}_n$ and $\bar{x}^h$ are the coordinates of its image under the infinitesimal transformation, ε is an infinitesimal parameter not depending on x^h, and ξ^h is a displacement vector, see [9].[10]

If a certain object $\mathcal{A}$ of the space $\mathbb{V}_n$ depends on $x \in \mathbb{V}_n$ but also on the infinitesimal parameter ε, then the *principal part* of the object $\mathcal{A}$ is $\overset{0}{\mathcal{A}}(x)+\overset{1}{\mathcal{A}}(x)\varepsilon$ in the expansion of series with respect to the infinitesimal parameter ε

$$\mathcal{A}(x,\varepsilon) = \overset{0}{\mathcal{A}}(x) + \overset{1}{\mathcal{A}}(x)\varepsilon + \overset{2}{\mathcal{A}}(x)\varepsilon^2 + \dots \quad .$$

[10] Mikeš J. et al., *Differential geometry of special mappings*, Palacky Univ. Press, Olomouc, 2015.

Infinitesimal rotary transformations

For our purposes the curves obtained by the infinitesimal transformation of geodesics satisfy the equations of isoperimetric extremals of rotation (1) under the condition, that we dropped the terms containing higher powers of the infinitesimal parameter ε, i.e. $\varepsilon^2, \varepsilon^3, \dots$.

Definition

An infinitesimal transformation of the (pseudo-) Riemannian space $\mathbb{V}_2$ is called *rotary* if it maps any geodesic of the space $\mathbb{V}_2$ onto an isoperimetric extremal of rotation in their principal parts.

We prove the following theorem.

Theorem

A differential operator $X = \xi^\alpha(x)\partial_\alpha$ $(\partial_\alpha = \partial/\partial x^\alpha)$ determines an infinitesimal rotary transformation of (pseudo-) Riemannian space $\mathbb{V}_2$ if and only if X satisfies

$$\mathcal{L}_\xi \Gamma^h_{ij} = \delta^h_{(i}\psi_{j)} + \theta^h g_{ij}, \quad \theta^h_{,i} = \theta^h(\theta_i + K_i/K) + \nu\delta^h_i, \tag{15}$$

where ψ_i is a covector, δ^h_i is the Kronecker delta, θ^h is a vector field, g is a metric tensor, K $(\neq 0)$ is the Gaussian curvature, and $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ.

As it can be seen, the equations (15) have simpler form than the equations of rotary trasnformations deduced by Leiko.[11]

[11] Leiko, S. G. Infinitesimal rotational transformations and deformations of surfaces in Euclidean space. Dokl. Math. 52, No. 2, 190-192 (1995).

Thank you for your attention.

References I

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