FANTASTIC SYMMETRIES AND WHERE TO FIND THEM

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Lecture 5: Heir-equations for evolution systems

- Classical Lie symmetries of partial differential equations: an example.
- Nonclassical symmetries of partial differential equations: an example.
- Iteration of the nonclassical symmetry method: heir-equations.
- Conditional Lie-Bäcklund symmetries and heir-equations.
- Nonclassical symmetries as special solutions of heir-equations.
- More symmetry solutions than expected with heir-equations.

Classical symmetries

Evolution equation:

$$u_t \;=\; \mathrm{H}\left(t, x, u, u_x, u_{xx}, \ldots, u_{\underbrace{xx\cdots}_{n}}\right)$$

Lie symmetry operator:

$$\Gamma = V_1(t,x,u)\,\partial_t \;+\; V_2(t,x,u)\,\partial_x \;+\; G(t,x,u)\,\partial_u$$

Determining equation:

$$\underset{n}{\Gamma}(\mathrm{u_t} - \mathrm{H})\Big|_{\{u_t - H = 0\}} = 0$$

Invariant surface:

$$V_1\, u_t \;+\; V_2\, u_x \;-\; G \;=0$$

An example

MCN, Atti Sem. Mat. Fis. Univ. Modena (1984)

In this paper we consider the flow of a viscous, homogeneous, incompressible fluid of finite electrical conductivity. The corresponding M.H.D. equations are:

$$\varrho(u_t + uu_x + vu_y + wu_z) - \eta(u_{xx} + u_{yy} + u_{zz}) + p_x + \{s(s_x - k_y) - r(k_z - r_x)\}/4\pi\mu = 0\,, \tag{1.1}$$

$$\varrho(v_t + uv_x + vv_y + wv_z) - \eta(v_{xx} + v_{yy} + v_{zz}) + p_y + \{r(r_y - s_z) - k(s_x - k_y)\}/4\pi\mu = 0\,, \tag{1.2}$$

$$\varrho(w_t + uw_x + vw_y + ww_z) - \eta(w_{xx} + w_{yy} + w_{zz}) + p_z + \{k(k_z - r_x) - s(r_y - s_z)\}/4\pi\mu = 0\,, \tag{1.3}$$

$$u_x + v_y + w_z = 0\,, \tag{1.4}$$

$$\left.\begin{aligned}
&(1.5)\quad k_t - (us - vk)_y + (wk - ur)_z - \nu_m(k_{xx} + k_{yy} + k_{zz}) = 0\,,\\
&(1.6)\quad s_t - (vr - ws)_z + (us - vk)_x - \nu_m(s_{xx} + s_{yy} + s_{zz}) = 0\,,\\
&(1.7)\quad r_t - (wk - ur)_x + (vr - ws)_y - \nu_m(r_{xx} + r_{yy} + r_{zz}) = 0\,,
\end{aligned}\right\}\quad (\nu_m = c^2/4\pi\mu\sigma_c)\,,$$

$$k_x + s_y + r_z = 0\,, \tag{1.8}$$

THEOREM. The full Lie group which leaves the M.D.H. equations (1.1-1.8) invariant is given by (2.1) with

(2.2) $$T = \alpha + 2\beta t\,,$$

(2.3) $$X = \beta x - \gamma y - \lambda z + f(t)\,,$$

(2.4) $$Y = \beta y + \gamma x - \sigma z + g(t)\,,$$

(2.5) $$Z = \beta z + \lambda x + \sigma y + h(t)\,,$$

(2.6) $$U = -\beta u - \gamma v - \lambda w + f'(t)\,,$$

(2.7) $$V = -\beta v + \gamma u - \sigma w + g'(t)\,,$$

(2.8) $$W = -\beta w + \lambda u + \sigma v + h'(t)\,,$$

(2.9) $$P = -2\beta p + j(t) - \varrho x f''(t) - \varrho y g''(t) - \varrho z h''(t)\,,$$

(2.10) $$K = -\beta k - \gamma s - \lambda r\,,$$

(2.11) $$S = -\beta s + \gamma k - \sigma r\,,$$

(2.12) $$R = -\beta r + \lambda k + \sigma s\,,$$

where α, β, γ, λ and σ are five arbitrary parameters and $f(t)$, $g(t)$, $h(t)$ and $j(t)$ are arbitrary, sufficiently smooth, functions of t.

Nonclassical symmetries

Introduced 50 years ago in a seminal paper [Bluman & Cole, J. Math. Mech., 1969] to obtain new exact solutions of the linear heat equation.

Determining equation:

$$\underset{n}{\Gamma}(\mathrm{u_t} - \mathrm{H})\Big|_{\left\{ \begin{array}{rcl} u_t - H & = & 0 \\ V_1 u_t + V_2 u_x - G & = & 0 \end{array} \right\}} = 0$$

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Nonclassical symmetries also called Q-conditional symmetries of second-type in Fushchych et al, 1993, or reduction operators in Popovych, J. Phys. A: Math. Theor., 2008.

Also a particular instance of the more general differential constraint method that, as stated in Kruglikov, Acta Appl. Math. 2008 *dates back at least to the time of Lagrange... and was introduced into practice by Yanenko* in 1961. The method was set forth in details in Yanenko's monograph Sidorov, Shapeev, Yanenko, 1984 that was not published until after his death.

Iterating NCSM

MCN, Phys. D (1994)

$$V_1(t,x,u)u_t + V_2(t,x,u)u_x = G(t,x,u)$$

$$V_1 = 0,\quad V_2 = 1 \Rightarrow u_x = \ G(t,x,u) \Rightarrow G - \textbf{equation}$$

$$\xi_1(t,x,u,G)G_t + \xi_2(t,x,u,G)G_x + \xi_3(t,x,u,G)G_u = \eta(t,x,u,G)$$

$$\xi_1 = 0,\ \xi_2 = 1,\ \xi_3 = G \Rightarrow G_x + GG_u = u_{xx} = \eta(t,x,u,G)$$

$$\Downarrow$$

η − **equation**

$$\eta_x + G\eta_u + \eta\eta_G = u_{xxx} = \Omega(t,x,u,G,\eta)$$

$$\Downarrow$$

Ω − **equation**

$$\Omega_x + G\Omega_u + \Omega\Omega_G + \Omega\Omega_\eta = u_{xxxx} = \rho(t,x,u,G,\eta,\Omega)$$

$$\Downarrow$$

ρ − **equation**

$$\vdots$$

Heir − equations

Heir-equations

Definition: Hierarchy of equations which admit the same Lie symmetry algebra (heirs) as the original one.
Each equation has one more additional independent variable than the previous equation in the hierarchy, and thus
WE CAN GET MORE SOLUTIONS FROM THE SAME SYMMETRY.

MCN, Physica D 78 (1994),
MCN, J. Phys. A: Math. Gen. 29 (1996)

CLASSICAL vs. NONCLASSICAL

BOTH SYMMETRIES ARE PARTICULAR SOLUTIONS OF THE SAME HEIR-EQUATION.

MCN, J. Math. Anal. Appl. 279 (2003)

Outline of the method

Second Order:

$$u_t = u_{xx} + H_1(t, x, u, u_x)$$

Invariant surface condition:

$$V_1(t, x, u)u_t + V_2(t, x, u)u_x = F(t, x, u)$$

$$V_1 = 1 \Rightarrow u_t + V_2(t, x, u)u_x = F(t, x, u)$$

$$u_{xx} + H_1(t, x, u, u_x) + V_2(t, x, u)u_x = F(t, x, u)$$

$$\text{i.e.} \qquad \eta = F(t, x, u) - V_2(t, x, u)G - H_1(t, x, u, G) \quad (*)$$

Generate the η-equation and search for the particular solution $(*)$.

Example:

$$u_t = u_{xx} + uu_x$$

yields

$$\eta = F(t, x, u) - V_2(t, x, u)G - uG.$$

Blow-up solutions

We recall Galaktionov's equation Diff. Int. Eqns., 1990:

$$u_t = u_{xx} + u_x^2 + u^2. \tag{1}$$

Its G-equation is:

$$2GG_{xu} + G^2G_{uu} + G^2G_u - u^2G_u - G_t + G_{xx} + 2GG_x + 2uG = 0. \tag{2}$$

Its η-equation is:

$$\begin{aligned} &2\eta\eta_{xG} + 2G\eta\eta_{uG} + \eta^2\eta_{GG} - 2uG\eta_G + 2G\eta_{xu} + \eta_{xx} \\ &+2G\eta_x - \eta_t + G^2\eta_{uu} + G^2\eta_u - u^2\eta_u + 2\eta^2 + 2u\eta + 2G^2 = 0. \end{aligned} \tag{3}$$

Lie symmetries: $X_1 = \partial_t$, and $X_2 = \partial_x$. Search for t-independent invariant solutions of (3): $\eta = \eta(x, u, G)$. A particular case is $\eta_u = 0 \Rightarrow \eta = L(x, G)$. Substituting this expression for η into (3) leads to $L = f(x)G$ with

$$f(x) = \frac{-c_1 \sin x + c_2 \cos x}{c_2 \sin x + c_1 \cos x}. \tag{4}$$

If we let $c_1 = 0$, then:

$$\eta = \cot(x)G, \tag{5}$$

which is just the differential constraint for (1) given in Olver, Proc. R. Soc. Lond. A, 1994 i.e.:

$$u_{xx} = \cot(x)u_x. \tag{6}$$

Integrating (6) with respect to x gives raise to:

$$u = w_1(t)\cos(x) + w_2(t). \tag{7}$$

Finally, the substitution of (7) into (1) leads to:

$$\dot{w}_1 = w_1^2 + w_2^2, \qquad \dot{w}_2 = 2w_1w_2 - w_2 \tag{8}$$

This is the solution derived by Galaktionov for (1).

A class of reaction-diffusion equations $u_t = u_{xx} + cu_x + R(u, x)$

M.S.Hashemi and MCN, J. Nonlinear Math. Phys. 20 (2013)

$$u_t = u_{xx} + cu_x - \frac{1}{2}x^2u^3 + 3u^2 + \frac{1}{2}c^2u$$

$$u(t,x) = \frac{c^2c_1e^{c^2t+\frac{cx}{2}} - c^2(1+cx)e^{\frac{-cx}{2}}}{c_2e^{\frac{-c^2t}{4}} + c_1(cx-2)e^{c^2t+\frac{cx}{2}} + (10+5cx+c^2x^2)e^{\frac{-cx}{2}}}$$

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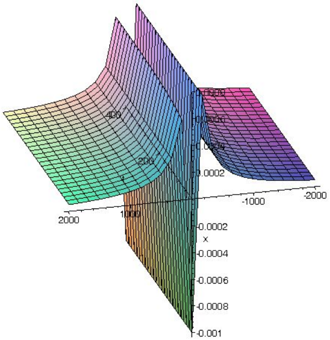
$[c = 0.1, c_1 = 2/10^5, c_2 = 0]$



$$u_t = u_{xx} + cu_x - \frac{1}{2}e^{cx}u^3 + \frac{c^2}{4}u + e^{\frac{cx}{2}}$$

$$u(t,x) = \frac{\sqrt[3]{2}}{2\left(-R_2(t)\sin\left(\sqrt[3]{2}\sqrt{3}x/4\right) + \cos\left(\sqrt[3]{2}\sqrt{3}x/4\right)\right)e^{cx/2}} \times$$

$$\times\left[+R_2(t)\left(\sin\left(\sqrt[3]{2}\sqrt{3}x/4\right) - \sqrt{3}\cos\left(\sqrt[3]{2}\sqrt{3}x/4\right)\right)\right.$$

$$\left.-\sqrt{3}\sin\left(\sqrt[3]{2}\sqrt{3}x/4\right) - \cos\left(\sqrt[3]{2}\sqrt{3}x/4\right)\right]$$

$$R_2(t) = -\tan\left(3^{3/2}2^{-7/3}t\right)$$

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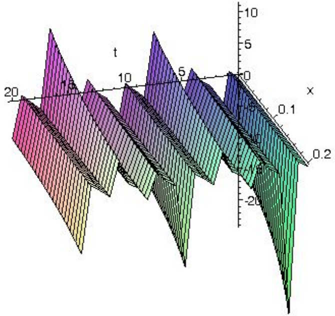
$$R_2(t) = -\tan\left(3^{3/2}2^{-7/3}t\right)$$

$[c = 2]$



Heir eqs for systems of PDE

MCN & B.Hajek, 2019

Outline of the method

Second Order:

$$\mathbf{u}_t = \mathbf{H}(t, x, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx})$$

Invariant surface conditions:

$$V_1(t, x, \mathbf{u})(u_j)_t + V_2(t, x, \mathbf{u})(u_j)_x = F_j(t, x, \mathbf{u}), \qquad (j = 1, \dots, n)$$

$$V_1 = 1 \Rightarrow (u_j)_t + V_2(t, x, \mathbf{u})(u_j)_x = F_j(t, x, \mathbf{u})$$

$$(u_j)_{xx} + \tilde{H}_j(t, x, \mathbf{u}, \mathbf{u}_x) + V_2(t, x, \mathbf{u})(u_j)_x = F_j(t, x, \mathbf{u})$$

$$\text{i.e.} \qquad \eta_j = F_j(t, x, \mathbf{u}) - V_2(t, x, \mathbf{u})G_j - \tilde{H}_j(t, x, \mathbf{u}, \mathbf{G}) \quad (*)$$

Generate the η_j-equation and search for the particular solution $(*)$.

Example of a system of two diffusion eqs

King, Proc.Roy.Soc.London A, 1990

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= u_2 \frac{\partial^2 u_1}{\partial x^2} - u_1 \frac{\partial^2 u_2}{\partial x^2}, \\ \frac{\partial u_2}{\partial t} &= \frac{\partial^2 u_1}{\partial x^2}. \end{aligned}$$

We generated the G_i and the η_i eqs, and then searched for the following particular solutions of the η_i eqs:

$$\eta_1 = F_2(t, x, u_1, u_2) - V_2(t, x, u_1, u_2) G_2,$$

$$\eta_2 = \frac{1}{u_1} \left(u_2 \eta_1 - F_1(t, x, u_1, u_2) + V_2(t, x, u_1, u_2) G_1 \right).$$

We obtained the classical symmetries

$$\eta_1 \equiv \frac{\partial^2 u_1}{\partial x^2} = a_2 u_1 + a_1, \quad \eta_2 \equiv \frac{\partial^2 u_2}{\partial x^2} = u_2 a_2 - a_3,$$

and also a nonclassical one...

Nonclassical symmetries:

$$\eta_1 = F_2(t, x, u_1, u_2) - V_2(t, x, u_1, u_2)G_2,$$

$$\eta_2 = \frac{1}{u_1}\left(u_2\eta_1 - F_1(t, x, u_1, u_2) + V_2(t, x, u_1, u_2)G_1\right).$$

with

$$V_2 = 0, \quad F_1 = -\frac{(a_3 \exp[(t + a_4)/a_3] - 2)u_1}{(a_3 \exp[(t + a_4)/a_3] + 2)a_3},$$

$$F_2 = \frac{a_1 u_1 \left(a_3 \exp[(t + a_4)/a_3] + 2\right)^2 + 4u_2 \exp[(t + a_4)/a_3]}{a_3^2 \exp[2(t + a_4)/a_3] - 4}.$$

and consequently a fourth-order equation in u_1 with respect to x is obtained.

More details in *MCN & B.Hajek, 2019*.

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Twenty years later, in 1998, Marcos Moshinsky was awarded the Wigner medal.

The effectiveness of Lie symmetries

Eugene P. Wigner, The unreasonable effectiveness of Mathematics in the Natural Sciences, *Comm. Pure Appl. Math 13* (1960) 1-14

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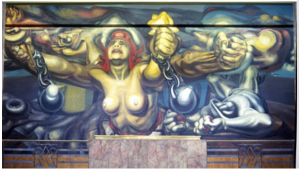
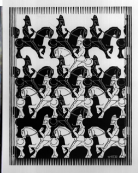
Paraphrasing Eugene Paul Wigner:

- **Lie symmetries turn up in entirely unexpected connection**
- **Lie symmetries permit unexpectedly accurate description of the phenomena in this connection**
- **A theory formulated in terms of Lie symmetries maybe uniquely appropriate.**

Marcos Moshinsky, **SIMETRÍA EN LA NATURALEZA**, Conferencia Inaugural En El Colegio Nacional (1972)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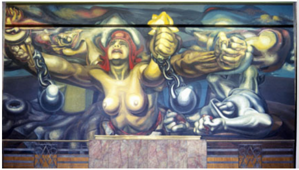

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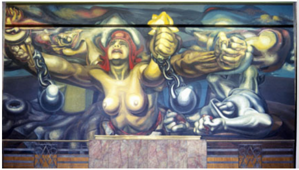

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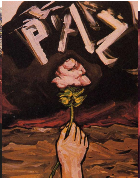

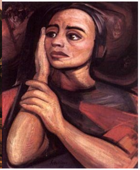

Marcos Moshinsky (1999) **Vine, vi y comprendí**